An Efficient Method for Market Risk Management under Multivariate Extreme Value Theory Approach

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Outline

Market Risk Assessment

Typically relies on quantile-based measures of risk

- Value at Risk (VaR)
- Expected Shortfall (ES)

Conventional methods of VaR and ES estimation in practice:

- Historical simulation
- Analytical method
- Monte Carlo simulation

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Interior vs. Tails

Problem with conventional methods:

- Single parametric family fitted to the whole distribution.
- Trying to reconstruct the entire distribution of returns.
- However, extremely large (but rare) losses are the ones that matter the most!

Extreme Value Theory (EVT) approach:

- Characterizes the tail behavior of distribution of returns.
- Focuses on extreme losses rather than interior.

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Previous Empirical Results

VaR and ES estimates obtained by using EVT-based models outperform the ones based on standard methods:

- McNeil (1997)
- Nyström & Skoglund (2002)
- Harmantzis, Chien & Miao (2005)
- Marinelli, d'Addona & Rachev (2007)
- Lai & Wu (2010)

Univariate EVT

Theorem

Picklands (1975): *Let* $\{X_i\}_{i=1}^n$ *be a set of n independent and identically distributed random variables with distribution function* F*, and* F_u *the distribution of excesses of* X *over the treshold* u*. Let* x_F *be the end of the upper tail of* F *(possibly* $+\infty$*). Then, there are constants* $\xi \in \mathbb{R}$ *and* $\beta \in \mathbb{R}_+$ *such that*

$$\lim_{u \to x_F} \sup_{u < X < x_F} |F_u(X) - G_{\xi,\beta}(X - u)| = 0,$$

where

$$G_{\xi,\beta}(y) := 1 - \left(1 + \xi \frac{y}{\beta}\right)_+^{-1/\xi}$$

is known as the generalized Pareto (GP) distribution.

Multivariate EVT?

Problems with univariate EVT:

- Useful for portfolio-level analysis only.
- Cannot capture behavior of correlations during extreme market movements.

Problems with available multivariate alternatives:

- Copulas
  - Too complicated for practitioners.
  - More difficult to implement than EVT.
  - Introduce additional ”model risk”.
- Multidimensional limiting relations
  - See e.g. Smith (2000) or Dupuis & Jones (2007).
  - Model complexity increases greatly with the number of risk factors.

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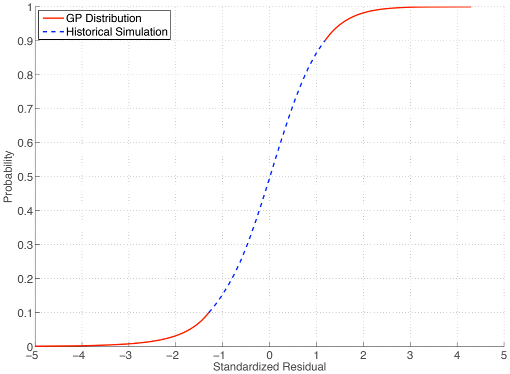
Basic Idea

- Use series of returns on individual assets in portfolio.
- Construct n orthogonal series that are approximately i.i.d.
- Use separate estimations of univariate EVT.

Estimation of the Univariate EVT

- Choose a reasonable treshold u.
- Use one of the conventional methods to fit the interior of the distribution (e.g. historical simulation).
- Fit the upper and lower tail separately with the GP distribution.

Estimation of the Univariate EVT



Principal Components

- In n dimensions, use the principal components of the unconditional VCV matrix:

$$
\begin{aligned}
\boldsymbol{\Lambda} &:= \operatorname{diag}(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n) \\
\mathbf{V}_\infty &= \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}' \\
\mathbf{L} &:= \mathbf{P}\boldsymbol{\Lambda}^{1/2} \\
\mathbf{z}_t &= \mathbf{L}^{-1}\varepsilon_t \\
\mathbb{E}\left(\mathbf{z}_t\right) &= \mathbf{0} \\
\operatorname{var}\left(\mathbf{z}_t\right) &= \mathbf{1}_n
\end{aligned}
$$

- This gives a set of (at most) n orthogonal standardized coordinates.

GARCH Filtering

- Conditional variance: GJR-GARCH(p, q)

$$
\begin{aligned}
\mathbf{V}_t &= \Omega + \sum_{s=1}^{p} \mathbf{A}_s \mathbf{E}_{t-s} + \sum_{s=1}^{p} \Theta_s \mathbf{I}_{t-s} \mathbf{E}_{t-s} + \sum_{s=1}^{q} \mathbf{B}_s \mathbf{V}_{t-s} \\
\mathbf{E}_t &:= \varepsilon_t \varepsilon_t' \\
\mathbf{I}_t &:= \mathrm{diag}(\mathrm{sgn}(-\varepsilon_{t,1})_+,\ \mathrm{sgn}(-\varepsilon_{t,2})_+,\ \ldots,\ \mathrm{sgn}(-\varepsilon_{t,n})_+)
\end{aligned}
$$

- Make transformation from returns to principal components.
- Estimate n separate univariate GJR-GARCH(p, q) models:

$$
\widehat{V}_{t,i} = \widehat{\Omega}_i + \sum_{s=1}^{p} \widehat{A}_{s,i} \widehat{E}_{t-s,i} + \sum_{s=1}^{p} \widehat{\Theta}_{s,i} \widehat{I}_{t-s,i} \widehat{E}_{t-s,i} + \sum_{s=1}^{q} \widehat{B}_{s,i} \widehat{V}_{t-s,i}
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Forecasting

- Confidence interval for the value of i-th principal component, h steps ahead:

$$z^{\pm}_{t+h,i} = F_i^{-1}(q_{\pm})\sqrt{\widehat{V}_{t+h,i}}$$

- Forecast of multivariate VaR:

$$\begin{aligned}
\mathbf{S}^{\pm}_{t'} &:= \operatorname{diag}\left[(z^{\pm}_{t',1})^2 \; (z^{\pm}_{t',2})^2 \; \ldots \; (z^{\pm}_{t',m})^2\right] \\
\mathbf{Q}^{\pm}_{t'} &:= \mathbf{L}\mathbf{S}^{\pm}_{t'}\mathbf{L}' \\
\mathrm{VaR}^{\pm} &= \mathbf{a}'\mu_{t'} \pm \sqrt{\mathbf{a}'\mathbf{Q}^{\pm}_{t'}\mathbf{a}}
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- For ES just replace F_i^{-1} by $(F_i^{-1} + \beta_{\pm} - \xi_{\pm}u_{\pm})/(1-\xi_{\pm})$.

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Data

- Daily returns computed from adjusted prices of Dow Jones stocks.
- DJIA composition on December 31, 2010 was used to choose the 30 constituents of the equally-weighted portfolio.
- 2401 observations in each of the series (June 14, 2001 – December 31, 2010).

Data


Daily returns (percent)
10
5
0
−5
−10
2001
2002
2003
2004
2005
2006
2007
2008
2009
2010

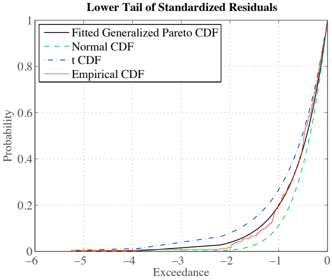
Parameters of Univariate GP Distributions

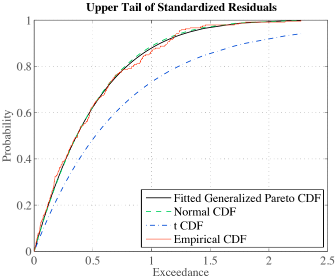
December 31, 2010

Parameter	PC 1	PC 2	PC 3	PC 4
$\widehat{\xi}_+$	-0.1458	0.0369	0.1179	0.0829
	(0.0588)	(0.0703)	(0.0719)	(0.0564)
$\widehat{\beta}_+$	0.5492	0.6055	0.4755	0.5598
	(0.0479)	(0.0582)	(0.0461)	(0.0481)
$\widehat{\xi}_-$	-0.0089	0.0859	0.0670	0.0207
	(0.0482)	(0.0679)	(0.0669)	(0.0701)
$\widehat{\beta}_-$	0.6222	0.5080	0.5917	0.6354
	(0.0503)	(0.0478)	(0.0553)	(0.0609)

Fitting the Tails

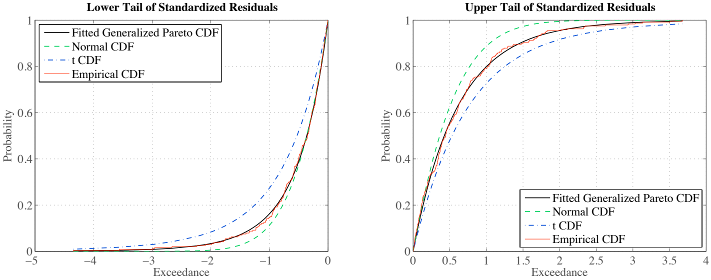
First principal component





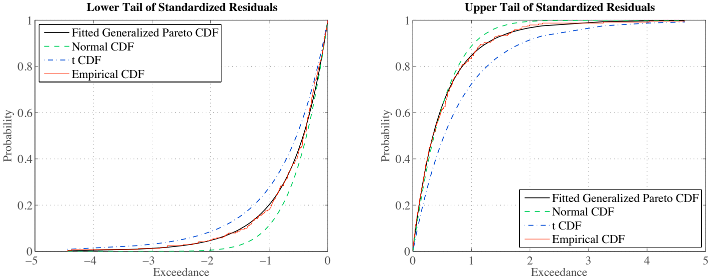
Fitting the Tails

Second principal component



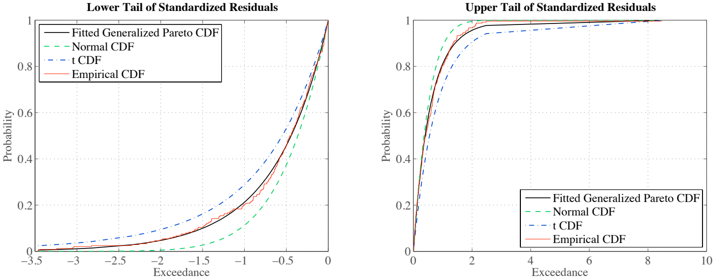
Fitting the Tails

Third principal component



Fitting the Tails

Fourth principal component



VaR and ES Forecasts

December 31, 2010

Upper tail					
CL	0.900	0.950	0.990	0.995	0.999
VaR	0.6431	0.8288	1.1975	1.3334	1.6058
ES	0.8901	1.0534	1.3789	1.4994	1.7428

Lower tail					
CL	0.900	0.950	0.990	0.995	0.999
VaR	−0.6173	−0.8366	−1.3416	−1.5574	−2.0546
ES	−0.9322	−1.1500	−1.6515	−1.8659	−2.3601

Backtesting Setup

- Last 1000 observations are chosen as out-of-the-sample data.
- Expanding-window estimation of the model for each day.
- Tomorrow's VaR forecasts compared with tomorrow's actual return on the portfolio.

Number of VaR Violations by Quantile: Upper Tail

Method	Number of violations				
CL	0.900	0.950	0.990	0.995	0.999
HS	374	307	195	143	89
mv Normal	97	42	13	5	2
mv Student	106	42	9	5	0
mv EVT	104	45	13	6	1
Expected	100	50	10	5	1

Number of VaR Violations by Quantile: Lower Tail

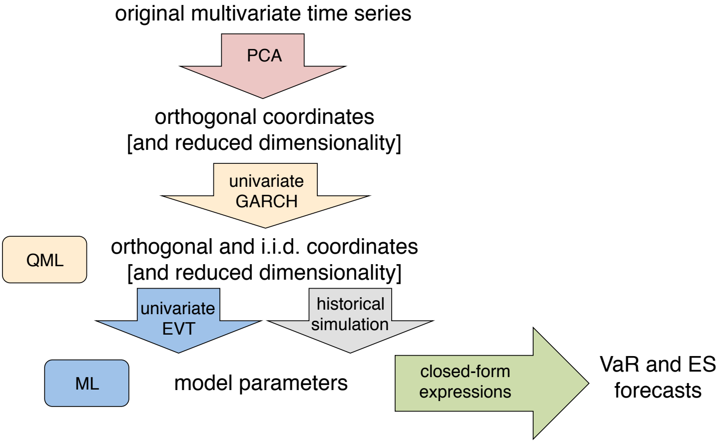
Method	Number of violations				
CL	0.900	0.950	0.990	0.995	0.999
HS	321	279	202	163	108
mv Normal	106	68	26	12	5
mv Student	105	68	23	11	4
mv EVT	110	68	13	5	1
Expected	100	50	10	5	1

Pearson's Test

Method	Lower tail	Upper tail
HS	12420.2	9029.60
	(0.0000)	(0.0000)
mv Normal	37.4700	6.5850
	(0.0000)	(0.2534)
mv Student	25.0828	6.6350
	(0.0001)	(0.2492)
mv EVT	8.8161	4.2878
	(0.1166)	(0.5088)

(p-values in parentheses.)

Summary of the Method



Conclusion

- The proposed approach employs the notion that some key results of the univariate EVT can be applied separately to a **set of orthogonal i.i.d. random variables**.
- Such random variables can be constructed from the **principal components of GARCH conditional residuals** of a multivariate return series.
- The approach yields **more precise VaR and ES forecasts** than conventional methods based on historical simulation, conditional normality or conditional t-distribution, without losing efficiency.