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# An Infinite-Dimensional Interest Rates Term Structure Model: Arbitrage-Free, Realistic and Practical

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Moscow, November, 2011

# The Problem

- In this presentation we focus on the zero-coupon yield curve **estimation** problem, no derivatives pricing (since there are none in Russia ☺).
- Term structure of interest rates can be considered for different market instruments: bonds, interest rate swaps, FRA, etc.
- These are **different** markets, and information from different markets should not be mixed for estimation of a single model.
- Here we consider only bond market as a source of information.

# Classification

- Term structure models as seen from the yield curve construction point of view:
- By information used:
  - Snapshot methods.
  - Dynamic methods.
- By a priori assumptions (later):
  - Parametric methods.
  - Nonparametric (spline) methods.

# Information Used

- **Static** methods use only a "market snapshot".
  - Bootstrapping
  - Parametric methods (Nelson-Siegel, Svensson).
  - Spline methods (Vasicek-Fong, Sinusoidal-Exponential Splines).
- **Dynamic** methods use time series data.
  - General affine term structure models.
  - HJM evolution of forward rates.
  - …

# The Data

- Available data: bonds, their prices, possibly bid-ask quotes, volumes, etc.
- NO interest rate derivatives are available on Russian market.
- Difficulties with observed data:
  - coupon-bearing bonds;
  - few traded bonds;
  - different credit quality and liquidity.

# Convenient compounding convention

- Continuous compounding:

$$d(t) = \exp[-t\, y(t)]$$

- Instantaneous forward rates:

$$y(t) = \frac{1}{t}\int_0^t r(t)dt$$

- This is mainly for simplicity reasons. Later we may introduce discrete compounding.

# Snapshot (static) fitting

# Data treatment

- Bond prices at the given moment.
- Bid/Ask quotes at the given moment.
- Other parameters: volumes, frequencies etc.
- Bond price is assumed to be approximately equal to present value of promised cash flows:

$$P_k \approx \sum_{i=1}^{n} d(t_i) F_{i,k}$$

# Different Approaches

- To solve a problem with insufficient data we have to introduce additional assumptions. Depending on these, we classify the approaches.
  - Assumptions on a specific parametric form of the yield curve (parametric methods).
  - Assumptions on the degree of the smoothness (in some sense) of the yield curve (spline methods).

# Parametric methods of yield curve fitting

Svensson ( 6 parameters) $\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2$

**Instantaneous forward rate is assumed to have the following form:**

$$\beta_0 + \beta_1 e^{-\frac{t}{\tau_1}} + \beta_2 [\frac{t}{\tau_1} e^{-\frac{t}{\tau_1}}] + \beta_3 [\frac{t}{\tau_2} e^{-\frac{t}{\tau_2}}]$$

Nelson-Siegel (4 parameters) is a special case of Svensson with $\beta_3 = \tau_2 = 0$

**Assuming specific functional form for yield curve is arbitrary and has no economic grounds**

# Nonparametric methods

- Usually splines.
- Flexibility.
- Sensibility.
- Possibility of smoothness/accuracy control.

# Requirements

1. Approximate discount function should be decreasing (i.e. forward rates should be non-negative) with initial value equal to one, and positive.
2. Approximate discount function should be sufficiently smooth.
3. Corresponding residual with respect to observed bond prices should be reasonably small.
4. The market liquidity should be taken into account to determine the reasonable accuracy, e.g. the residual can be related to the size of bid-ask spreads.

# Problem statement ensuring positive forward fates

- Select the solution in the form:

$$d(t) = \exp\left[-\int_0^t f^2(\tau)\right] d\tau$$

- Introduce a penalty for non-smoothness (regularization) term:

$$\int_0^T f'^2(\tau)d\tau \to \min$$

- Minimize the residual with a regularization term added:

$$\sum_{k=1}^{N} w_k \left( \sum_{i=1}^{n} exp\left[-\int_0^{t_i} f^2(\tau)d\tau\right] F_{i,k} - P_k \right)^2 + \alpha \int_0^T f'^2(\tau)d\tau \to \min_f$$

# The semi-analytical solution

*Smirnov & Zakharov (2003):*

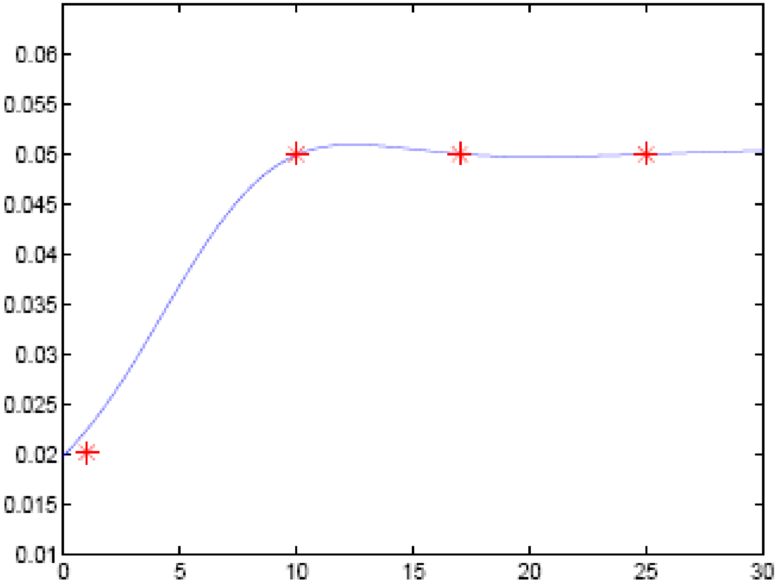
The optimal *f(t)* is a spline of the following form:

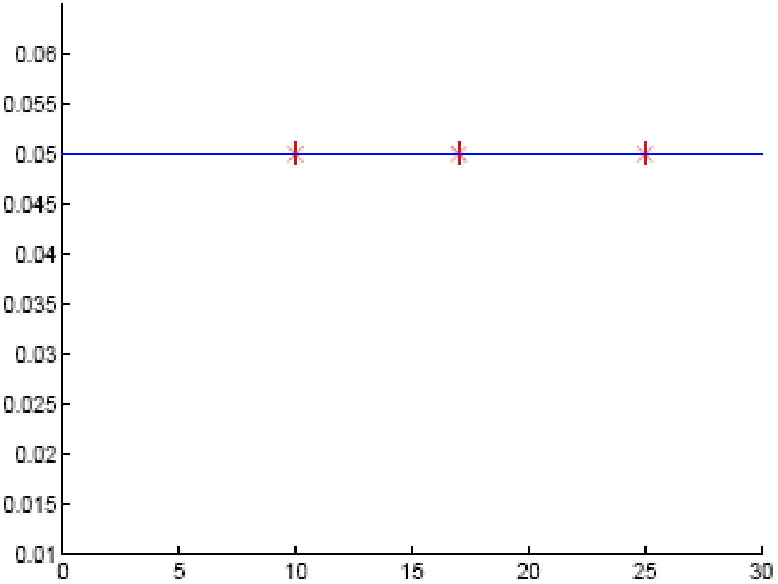
$$f(t) = \begin{cases} C_1 \exp\{\sqrt{\lambda_k}(t - t_{k-1})\} + C_2 \exp\{-\sqrt{\lambda_k}(t - t_{k-1})\}, & \lambda_k > 0 \\ C_1 \sin(\sqrt{-\lambda_k}(t - t_{k-1})) + C_2 \cos(\sqrt{-\lambda_k}(t - t_{k-1})), & \lambda_k < 0, \\ C_1(t - t_{k-1}) + C_2, & \lambda_k = 0 \end{cases}$$

$$t \in [t_{k-1}; t_k] \quad f(t_k - 0) = f(t_k + 0), \quad f'(t_k - 0) = f'(t_k + 0)$$

# History can be useful

- Consider to consecutive days, when short term bonds are not traded (or quoted) the second day:



# Stochastic Evolution

# Low-Dimensional Models

- Dynamics of several given variables (usually instantaneous rate and some others).
- Low-dimensional dynamic models imply non-realistic zero-coupon yield curves when calibrated to time series data and imply non-realistic dynamics when calibrated to snapshot data.

# Consistency Problems

- Very few static models may be embedded into a stochastic dynamic model in an arbitrage-free manner (Björk, Christensen, 1999; Filipović, 1999).
- Nelson-Siegel model allows arbitrage with every non-deterministic parameter dynamics.

$$r_t(x) = \beta_t^0 + \beta_t^1 e^{-\frac{x}{\tau_t}} + \beta_t^2 \frac{x}{\tau_t} e^{-\frac{x}{\tau_t}}$$

# Consistency Problems - II

- Nearly all arbitrage-free dynamic models are primitive.
- All such models are affine (Björk, Christensen, 2001; Filipović, Teichmann, 2004).

$$r_t(x) = h_0(x) + \sum_{i=1}^{N} Y_i(x)\lambda_{i,t}$$

# HJM-Style Models

- All modern HJM-style models are "whole yield curve models". And as such, they require that a whole yield curve be given as an input.
- But that is exactly our goal!
- We need a new model to fit our purpose.

# The Challenge

For term structure estimation:

- Snapshot methods
  - – Fail on illiquid markets with missing data.
  - – Don’t allow for dynamic extensions without introducing arbitrage opportunities.
- Dynamic models
  - – Exhibit awkward snapshot properties: unrealistic yield curves, etc., or
  - – Assume that a whole yield curve is observed.

# Goals

- Construction of an arbitrage-free nonparametric dynamic model, allowing for sensible snapshot zero-coupon yield curves.
- Peculiarities of data:
  - Incompleteness: only **several coupon-bearing** bonds are observed.
  - **Unreliability**: price data may be subject to errors and non-market issues.

# Our Solution

- We certainly need a dynamic model.
- Since finite-dimensional dynamic models suffer from the “curse of affinity”, we seek an infinite-dimensional model.
- We stress the estimation part and a model for observations.
- We also construct numerical algorithms and test the model on real data.

# Heath-Jarrow-Morton (1992) Approach

- Modeling all forward rates at once:

$$f(t,t') = f(0,t') + \int_0^t \alpha(u,t',\omega)du + \sum_{i=1}^{n} \int_0^t \sigma^i(u,t',\omega)dW_i(u),$$

$$0 \le t \le t' \le \tau.$$

- t – current time, t' – maturity time,
- Brace, Musiela (1994): One infinite-dimensional equation. $r_t(x) = f(t, x+t), \quad x,t \in \mathbb{R}_+$
- Filipović (1999): Infinite number of Brownian motions.
- Also being added: credit risks and stochastic volatility.

# The Model

- Based on the infinite-dimensional (Filipović,1999) extension of the HJM framework.
- In Musiela parameterization:

$$dr_t = (Dr_t + \alpha_t)dt + \sum_{j=1}^{\infty} \tilde{\sigma}_t^j \, d\beta_t^j$$

- No-arbitrage condition:

$$\alpha(x) = \sum_{j=1}^{\infty} \tilde{\sigma}^j(x) \int_0^x \tilde{\sigma}^j(\tau) d\tau$$

# The Simplest Possible Model

- Linear local volatility: $\tilde{s}^{j}(t,w,h)(x) = s^{j}(x)h(x).$
- Objective dynamics required for estimation from time series.
- Market price of risk is constant for each stochastic factor.
- Finite horizon:
  - Observations only up to a known T.
  - $r(x) = const \text{ for } x \geq T$
  - Realistic.

# Model Specification

- Itō SDE in Sobolev space $W_2^1[0,T]$.

$$dr_t = \left( Dr_t + \sum_{j=1}^{\infty} r_t \sigma^j I(r_t \sigma^j) - \sum_{j=1}^{\infty} r_t \sigma_t^j \gamma^j \right) dt + \sum_{j=1}^{\infty} r_t \sigma_t^j \beta_t^j ,$$

$$(I\ f)(x) = \int_0^x f(\tau) d\tau,$$

$\gamma^j -$ market price of risk for j$^{\text{th}}$ stochastic factor,

$\sigma^j(\cdot) -$ volatility parameters (functions).

# Observations

- Need a way to incorporate the stream of new information.
- Let $q_k(r)$ be the price of the k-th bond with respect to the true forward rate curve *r*.

$$q_k(r) = \sum_{s=0}^{n} F_{s,k} e^{-\int_0^{\tau_s} r(x)\mathrm{d}x}.$$

- Let the observed prices $p_k$ be random with distribution

$$p_k \sim N(q_k(r), w_k)$$

- $w_k$ is of order of the bid-ask spread (see next slide).

# Credibility

- Credibility is a degree of reliability of a piece of information (logical interpretation of probability).
- Standard deviation $w_k$ of the observation error is assumed to be directly dependent on the credibility.
- Factors affecting credibility:
  - Bid-ask spread.
  - Deal volume.
  - Any other factors.

# Regularization

- To estimate an infinite-dimensional entity from a finite number of discrete and error-prone observations requires regularization.
- We regularize the problem in two ways.
  - First, we impose a regularizing condition on observations (see next slide).
  - Second, our way of truncating the series when actually doing calculations is a kind of regularization.
- We choose both as to have economic meaning.

# Smoothness

- The yield curves used by market participants to determine the deal price are sufficiently smooth.
- Non-smoothness functional *J(r)* has to be chosen.
- Each observation is conditionally independent.
- Bayesian approach: conditional on observation $p^{(i)}$ at time $t_i$

$$\frac{dP_{r_{t_i}}}{dP_{r_{t_i-0}}} \propto N\left(q(r_{t_i}) - p^{(i)}, diag(w_k)\right) \cdot e^{-\alpha J(r)}.$$

# Snapshot Case

- We now examine snapshot properties of our model.
- Low-dimensional model fail to produce a plausible yield curve in a snapshot.
- Whole yield curve models fail to act as estimation models in a snapshot.
- HJM models are known to allow for a finite-dimensional parametric snapshot only in affine case.
- We have a new framework and a slightly different notion of a “snapshot”. And we don’t want a parametric snapshot model. We are non-parametric, so an infinite dimensional projection is just fine as long as it works.

# Snapshot Case

- Choose a special non-smoothness functional:

$$J(r) = \int_0^T \left( \frac{d\sqrt{r_t(\tau)}}{d\tau} \right)^2 d\tau$$

- Conditional on 1 observation: all prices observed at the same time (snapshot) and using flat priors:

$$-\log P(r) = \sum_{k=1}^{N} w_k^{-1} \left( q_k(r) - P_k \right)^2 + \alpha \int_0^T \left( \frac{d\sqrt{r(\tau)}}{d\tau} \right)^2 d\tau \rightarrow \min .$$

- This problem formulation leads to a known non-parametric model (see our snapshot model).

# Model Validation

- 3 time spans, Russian market, MICEX data, snapshots 3 times per day.
  - 10 jan 2006 – 14 apr 2006, normal market.
  - 1 aug 2007 – 28 sep 2007, early crisis.
  - 26 sep 2008 – 30 dec 2008, full crisis.
- In the normal market conditions the model is not rejected with 95% confidence level.
- Works reasonably on the crisis data.

# Complexity

- Market data is limited.
- Only enough to identify models with effective dimension = 2,3.
- More complex models are not identifiable.
- Tikhonov principle: the best model is the simplest one providing the acceptable accuracy.

# Main Results

- Arbitrage-free nonparametric dynamic yield curve construction model, providing:
  - Plausible and variable snapshot curves.
  - A good snapshot method as a special case.
  - Positive spot forward rates.
  - Liquidity consideration: inaccuracy and incompleteness in observations.
- Numerical algorithms and implementation tested on the real market data.

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# Thank you for your attention!

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