Uniqueness of Equilibrium in the Single-Period Kyle'85 Model

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Abstract

We analyze a one-period Kyle model (Kyle (1985)) where the risk-neutral informed trader can use arbitrary (linear or non-linear) deterministic strategies, and the market maker can use arbitrary pricing rules. We show that the standard linear insider's strategy, and correspondingly, the linear pricing rule, lead to the unique equilibrium in the model, even if the possible strategies are extended to arbitrary nonlinear piece-wise continuously differentiable functions of the fundamental. This means that there is a unique equilibrium in Kyle (1985), achieved on the standard linear insider's strategy, and the linear pricing rule.

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2 Introduction

We prove uniqueness of the linear equilibrium in the single-period trading model studied in Kyle (1985). Kyle (1985) examines a Nash equilibrium of a single-period trading game in which a monopolistic informed trader chooses a possibly non-linear trading strategy to maximize profits and competitive market makers simultaneously choose a possibly non-linear pricing rule. This pricing rule makes markets efficient in the sense of always generating zero expected profits for the market makers. Kyle (1985) shows that there is only one equilibrium in which the trading strategy and pricing rule are both happen to be linear functions. In this paper, we show that this equilibrium with linear trading strategies is unique within the class of piece-wise continuously differentiable functions; in this sense, we show that there does not exist an equilibrium with a non-linear trading strategy or pricing rule.

In our proof, we formulate the equilibrium problem as a fixed point problem of a particular functional, defined in an appropriate function space. We explicitly construct the pricing functional for the arbitrary insider's strategy, and study its analytic properties. Using these analytic properties, we derive the asymptotic form of both the pricing rule, and the insider's strategy at equilibrium. Taking into account the analytic properties of the pricing functional, and the above results, we conclude that the only possible equilibrium (under some technical conditions outlined below) is the one derived in Kyle (1985).

Rochet and Vila (1994) examine equilibria in a model where the informed trader observes both the liquidation value v and the noise trade u. Observing the level of noise trading is similar to being able to condition the quantity traded on price, as in the the “limit-order" model of Kyle (1989). The limit-order model is very different form the “market-order” setting of Kyle (1985), which we consider in this paper. In our setting, the informed trader does not observe the level of noise trading, as a result of which the analysis of our problem becomes much more complex.

3 Model and Assumptions

The model is the same as Kyle (1985), which we review briefly here. A single risk neutral informed trader, who recognizes that he has monopoly power, privately observes the realization v of the liquidation value of a risky asset drawn from a normal distribution with mean zero and variance σ_0^2. Price-taking liquidity traders trade a quantity denoted u, drawn independently from v from a normal distribution with mean zero and variance σ_u^2. After observing v, the informed trader chooses to trade a quantity x. x is a "market order," in the sense that it does not depend on the equilibrium price, P, but does depend on the observed value v. The informed trader does not observe the quantity of noise trading u prior to submitting his order. The price is set by market makers at which they trade the quantity necessary to clear the market. When doing so they observe the aggregate order flow $y = x + u$ but not x and u separately.

The Nash equilibrium studied in Kyle (1985) is formally defined as two functions, a trading strategy $X^*(\cdot)$ and a pricing rule $P^*(\cdot)$, satisfying a profit maximization condition and a market efficiency condition. The profit maximization condition states that the quantity traded by the informed trader, $x = X^*(\cdot)$, maximizes the informed trader's expected profits, taking the pricing rule $P^*(\cdot)$ as given, i.e.,

$$X^*(v) = \arg\max_x \mathrm{E}_u[(v - P^*(x+u))x|v]. \tag{1}$$

The market efficiency condition states that the market makers' expected profits equal to zero, conditional on observing the order flow, and taking the informed trader's trading strategy as given, i.e.,

$$P^*(y) = \mathrm{E}[v|X^*(v) + u = y]. \tag{2}$$

The proof that there exists a unique equilibrium in which $X^*(\cdot)$ and $P^*(\cdot)$ are linear functions is straightforward. Conjecture a linear trading strategy of the form $X^*(v) = \alpha + \beta v$ and a linear pricing rule of the form $P^*(y) = \mu + \lambda y$. Plugging the conjectured linear trading strategy into the profit maximization condition yields

$$X^*(v) = \arg\max_x \mathrm{E}_u[(v - \mu - \lambda(x+u))x|v] \quad \forall v \in (-\infty, +\infty). \tag{3}$$

Since $P^*(\cdot)$ is a linear function, the expectations operator has the effect of making the zero-mean noise trade u disappear from the maximization problem. Note that the noise trade u would not disappear in such a simple manner if the pricing rule were non-linear, an issue which becomes important when we examine whether non-linear equilibria exist. The informed trader's maximization problem becomes

$$X^*(v) = \arg\max_x [(v - \mu - \lambda x)x] \quad \forall v \in (-\infty, +\infty). \tag{4}$$

The solution to (4) is equal to

$$X^*(v) = \frac{v - \mu}{2\lambda}, \tag{5}$$

implying that α and β are given by

$$\alpha = \frac{\mu}{2\lambda}, \quad \beta = \frac{1}{2\lambda}. \tag{6}$$

Plugging the conjectured linear trading strategy into the pricing rule from equation (2) results in

$$P^*(y) = \mathrm{E}[v|\alpha + \beta v + u = y]. \tag{7}$$

Applying the projection theorem for normal random variables to the right-hand-side of equation (7) yields

$$P^*(y) = -\frac{\alpha\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} + \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} y. \tag{8}$$

The conditional expectation on the right-hand-side of equation (7) is a linear function of the order flow y because the trading strategy is linear and the random variables are normally distributed. If the trading strategy were non-linear, the right-hand-side would be a non-linear function of y. This becomes an important issue when non-linear equilibria are examined below. The solutions for μ and σ are given by

$$\mu = \frac{\alpha\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2}, \quad \lambda = \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2}. \tag{9}$$

Therefore, the unique solution of equations (6) and (9) for four parameters $\alpha, \beta, \mu, \sigma$ defining the equilibrium linear trading strategy and pricing rule is

$$\mu = \alpha = 0, \quad \beta = \frac{\sigma_u}{\sigma_v}, \quad \lambda = \frac{1}{2}\frac{\sigma_v}{\sigma_u}, \tag{10}$$

i.e., the linear equilibrium trading strategy and pricing rule are given by

$$X^*(v) = \frac{\sigma_u}{\sigma_v} v, \quad P^*(y) = \frac{1}{2}\frac{\sigma_v}{\sigma_u} y. \tag{11}$$

We refer to (11) as the *standard linear solution*, which is the same as in Kyle (1985). In what follows, we will assume without loss of generality that $\sigma_u = \sigma_v = 1$.

Note that this proof shows that a linear trading strategy implies a linear pricing rule and vice versa. Thus, in any equilibrium, *either* both the trading strategy and pricing rule are linear *or* neither the trading strategy nor pricing rule are linear. The above derivation of the unique *linear* equilibrium does not rule out the possibility of equilibria with non-linear trading strategies and corresponding non-linear pricing rules. Non-linear equilibria are ruled out by the analysis which follows in this paper.

To examine non-linear trading strategies and pricing rules, it is useful to have notation which describes the reaction function of the market makers to a possibly non-linear trading strategy of the informed trader. Such a reaction function is obtained by re-writing the pricing rule of equation (2) in the following manner to emphasize the functional dependence on the informed trader's trading strategy $X(\cdot)$:

$$P(y; X(\cdot)) = \mathrm{E}[v|X(v) + u = y]. \tag{12}$$

The notation $P(y; X(\cdot))$ indicates that the price depends on both a scalar argument given by the aggregate order flow y and a function argument given by the demand function $X(\cdot)$ that the market makers believe the informed trader is using. In what follows, we make extensive use of functionals, i.e. functions whose domain is a space of functions and whose range is a space of scalars. To keep notation clear, we follow (12) by placing scalar arguments in front of function arguments, separating the two types of arguments by a semi-colon, and using a dot to indicate that a function takes one scalar argument. For clarity, we also generally use lower-case letters to denote scalars and upper case letters to denote functions or functionals, except for pdf's, where we use lower case letters to avoid confusion with cdf's[1].

To facilitate the analysis which follows, we redefine the equilibrium concept as a fixed point problem using functionals. When the insider observes the realization v and trades quantity x, while the market makers believe that the insider follows the strategy $X_M(\cdot)$, then the insider's expected payoff, $\Pi(v, x; X_M(\cdot))$, is given by

$$\Pi(v, x; X_M(\cdot)) = \mathrm{E}_u[(v - P(x + u; X_M(\cdot)))x]. \tag{13}$$

When the insider follows the strategy $X_I(\cdot)$ and the market makers believe that the insider follows

[1] When we focus on the dependence of functionals on numerical arguments, we will use short hand notation and drop the functional arguments of the functionals. For example, the pricing rule will be viewed as a function of the order flow $P(\cdot)$. However, we will keep in mind that the functional form of the pricing rule depends on the functional form of the conjectured insider's strategy $X(\cdot)$.

the strategy $X_M(\cdot)$, then the insider's expected payoff, denoted $\overline{\Pi}(X_I(\cdot), X_M(\cdot))$, takes the form

$$\overline{\Pi}(X_I(\cdot), X_M(\cdot)) = \mathrm{E}_v[\Pi(v, X_I(v); X_M(\cdot))]. \tag{14}$$

The Bayesian Nash Equilibrium (BNE thereafter) strategy, $X_N^*(\cdot)$, is defined by the fixed-point condition

$$X_N^*(\cdot) = \arg\max_{X_I(\cdot)} \overline{\Pi}(X_I(\cdot), X_N^*(\cdot)). \tag{15}$$

This condition states that in a BNE, when the informed trader takes as given the trading rule the market makers believe he is going to follow, then the informed trader chooses the trading rule the market makers believe he is going to follow. The insider optimizes the expected payoff, since in a BNE he chooses the strategy $X_I(\cdot)$ *before* observing the fundamental v, and the expected payoff is given by (14). Although our reaction-function notation emphasizes the choice of the function $X_I(\cdot)$, the condition (15) leads to a definition of the Nash equilibrium conceptually equivalent to the one in Kyle (1985), defined above in equations (1) and (2). To see that the two definitions are equivalent, substitute the definition of $\overline{\Pi}$ into equation (15) to obtain

$$X_N^*(\cdot) = \arg\max_{X_I(\cdot)} \mathrm{E}_v[\Pi(v, X_I(v); X_N(\cdot))]. \tag{16}$$

The expectation on the right-and-side is maximized by solving the following point-by-point maximization problem:

$$X_N^*(v) = \arg\max_{x} \ \Pi(v, x; X_N(\cdot)) \quad \forall v \in (-\infty, +\infty). \tag{17}$$

From the definition of Π in equation (13), it follows that the maximization problem on the right-hand-side is the same as the Kyle (1985) profit maximization condition in equation (1). Furthermore, the fact that the strategy $X_N^*(\cdot)$ on the left-hand-side is the same as the $X_N(\cdot)$ on the right-hand-side insures that the Kyle (1985) pricing rule in equation (2) or equation (12) holds for all v.

In what follows we are going to restrict our analysis to the class of *admissible* strategies defined below.

Definition 1. *A trading strategy $X(\cdot)$ is admissible iff* $\mathrm{E}_v[(X(v))^2] < \infty$.

Using this definition, trading strategies which differ on sets of measure zero are considered to be identical. We believe that our uniqueness result would still follow if this assumption were relaxed to include measurable functions with infinite second moments. Restricting strategies to a version of L_2, however, makes our derivations more straightforward. We therefore make the following assumption.

Assumption 1. *Insider's optimal strategies are admissible.*

In what follows, we show that the strategies with infinitely increasing L_2 norm are suboptimal for the insider, in a sense that the insider's expected payoffs decrease when the L_2 norm of the strategy infinitely increases. This motivates the assumption that the equilibrium trading strategies have a finite L_2 norm.

4 First- and Second-Order Conditions

In this section, we use Bayes law to derive explicit expressions for the pricing rule P, introduce the expected price $\overline{P}$, show that P and $\overline{P}$ are analytic functions of their respective scalar arguments, examine first-order condition (FOC) and discuss local second-order condition (SOC).

Making use of the relations (12) and (25), we obtain the following characterization of the price functional through a straightforward application of Bayes law:

Lemma 1. *The price functional has the following characteristics:*

1. Let $f_{V,Y}$ denote the joint density of the liquidation value v and the total order flow y. For a given trading strategy $X(\cdot)$, the probability density $f_{V,Y}$ is given by

$$f_{V,Y}(v, y; X(\cdot)) = \frac{1}{2\pi} \exp\left[-\frac{(y - X(v))^2}{2}\right] \exp\left[-\frac{v^2}{2}\right]. \tag{18}$$

Let f_Y denote the marginal probability density of y given trading strategy $X(\cdot)$, defined by

$$f_Y(y; X(\cdot)) = \int_{-\infty}^{+\infty} dv f_{V,Y}(v, y; X(\cdot)). \tag{19}$$

Let g_Y denote the expectation over v defined by

$$g_Y(y; X(\cdot)) = \int_{-\infty}^{+\infty} dvv f_{V,Y}(v, y; X(\cdot)). \tag{20}$$

Then the pricing rule P is given by

$$P(y; X(\cdot)) = \frac{g_Y(y; X(\cdot))}{f_Y(y; X(\cdot))}. \tag{21}$$

2. Taking as given a trading strategy $X(\cdot)$, the pricing rule $P(y)$ is a meromorphic function of the order flow y, and can be expressed as converging series

$$P\left(y; X\left(\cdot\right)\right) = \sum_{k=0}^{+\infty} C_k\left(X\left(\cdot\right)\right)\ y^k, \tag{22}$$

where $\{C_k\left(X\left(\cdot\right)\right)\}$, $k = 0, 1, ..$are real functionals of the trading strategy $X\left(\cdot\right)$.

Proof: See Appendix.

Discussion of Proof: The proof utilizes the fact that the p.d.f. for the normal density not only is analytic but also has tails which die out quickly. Part 1 applies Bayes rule to the assumed normal distributions for the liquidation value v and the liquidity trading quantity u, expressing the pricing rule $P\left(y; X\left(\cdot\right)\right)$ as a ratio of two integrals. In the proof of Part 2, it is shown that, holding $X\left(\cdot\right)$ fixed, the integral in the numerator g_Y and the integral in the denominator f_Y are each *entire* functions of the order flow y. The pricing rule P is *meromorphic* as a ratio of two entire functions, and is analytic at all points such that the denominator does not vanish. Since the denominator g_Y is a p.d.f. which is strictly positive for all real arguments, the ratio P is analytic for all real values of the order flow y. Given a *linear* trading strategy $X(\cdot)$, we already know from Kyle (1985) that the pricing rule P is linear, and therefore a very simple kind of entire function. Thus, we already know that given a linear trading strategy $X(\cdot)$, the denominator f_Y does not vanish even when a complex argument is plugged into its power series expansion. Given an arbitrary *nonlinear* trading strategy $X(\cdot)$, the pricing rule P is not necessarily an entire function because the denominator may vanish for complex arguments $y = \{y_k\}$, $k = 0, 1, 2..$, when the real function P is extended to the complex plane using power series expansions. We follow the standard notation of Knopp (1996), and enumerate the poles according to their distance to the origin, so that $|y_0| \leq |y_1| \leq ...|y_k|...$

The series expansion (22) converges in a circle $|y| < R_c$ where the convergence radius R_c equals the modulus of the closest to the origin pole, $R_c = |y_0|$. An example is discussed in the Appendix.

For further analysis, we need to characterize the pricing rule in terms of uniformly converging series. This is provided by the following technical result[2].

Result 1. *Taking as given a trading strategy $X(\cdot)$, the pricing rule $P(y)$ can be expressed as uniformly converging series*

$$P(y) = \sum_{k=0}^{+\infty} h_k(y) + C_0 + C_1 y + C_2 y^2, \tag{23}$$

where $\{C_n\}$, $n = 0, 1, 2$ are real functionals of the trading strategy $X(\cdot)$ and

$$h_k(y) = \frac{A_k}{y - y_k} \left(\frac{y}{y_k} \right)^3, \tag{24}$$

with $\{A_k\}$ and $\{y_k\}$, $k = 0, 1, 2, ..$being complex valued functionals of $X(\cdot)$ representing the residues and poles of the meromorphic function P, respectively.

Proof: See Appendix.

Note that $\{h_k\}$, $k = 0, 1, 2, ..$ represent a singular part of the pricing rule P. These terms are absent if the pricing rule is an entire function, i.e. is represented by the power series converging in the entire complex plane. In particular, this is the case for the linear strategies. According to the Result 1, the pricing rule $P(y)$ is meromorphic in y and is therefore analytic in y in all complex plane, except for the isolated complex poles. Since $P(y)$ has no poles on the real axis, all the poles have a non-zero imaginary part. Note that the poles cannot have any finite accumulation point on the complex plane, i.e. they are isolated (Knopp, 1996). Since $P(y)$ is real for the real y, and has no poles on the real axis, all the poles $\{y_j, \quad k = 0, 1, 2..\}$ of $P(y)$ are complex, and come in pairs $\{y_k, y_k^*\}$, where y_k^* stands for the complex conjugate of y_k.

Now we define the expected price functional. Let $\overline{P}(x; X(\cdot))$ denote the expected price obtained by the informed trader when he trades quantity x and the market makers believe he is using the trading strategy $X(\cdot)$. Then $\overline{P}(x; X(\cdot))$ is a functional defined by

$$\overline{P}(x; X(\cdot)) = \mathrm{E}_u [P(x + u; X(\cdot))] . \tag{25}$$

[2] Here we use a short hand notation described in footnote 1.

Note that in equation (8), the right-hand-side defines a linear function of y in which the slope and intercept depend on the functional form of $X(\cdot)$ through the intercept parameter α and the slope parameter β in terms of which the linear strategy $X(v) = \alpha+\beta v$ is defined. If the reaction functional $P(y; X(\cdot))$ happens to be a polynomial function of y, then the coefficients of the polynomial will depend on the structure of some non-linear trading strategy $X(\cdot)$. More generally, if the functionals $P(y; X(\cdot))$ and $\overline{P}(x; X(\cdot))$ are analytic functions of y, each defined by its own power series, then the coefficients of each power series will depend in a particular way on the structure of the trading strategy $X(\cdot)$. Intuition suggests that since $P(y; X(\cdot))$ and $\overline{P}(x; X(\cdot))$ are defined by integrals which involve added noise, then $P(y; X(\cdot))$ and $\overline{P}(x; X(\cdot))$ should be relatively smooth functions of their scalar arguments y and x respectively, perhaps even analytic. We show below that this intuition turns out to be correct. Both $P(y; X(\cdot))$ and $\overline{P}(x; X(\cdot))$ are analytic functions of their scalar arguments, even when $X(\cdot)$ is an arbitrary piecewise continuously differentiable function which perhaps has discontinuities and is thus not itself analytic.

Based on the above definitions, we obtain the following first-order conditions for the informed trader's profit maximization problem, summarized by Proposition 1.

Proposition 1. *A necessary condition for a BNE is that for all admissible variations $\delta X(\cdot)$ we have*

$$\begin{aligned} 0 &= \delta_1 \overline{\Pi}\left(X_N^*(\cdot), \delta X(\cdot); X_N^*(\cdot)\right) \\ &= \mathrm{E}_v\left[\left\{v - \overline{P}\left(X_N^*(v); X_N^*(\cdot)\right) - X_N^*(v)\,\overline{P}'\left(X_N^*(v); X_N^*(\cdot)\right)\right\}\delta X(v)\right], \end{aligned} \tag{26}$$

and therefore

$$v - \overline{P}\left(X_N^*(v); X_N^*(\cdot)\right) - X_N^*(v)\,\overline{P}'\left(X_N^*(v); X_N^*(\cdot)\right) = 0. \tag{27}$$

Proof: Evaluating the first functional variation (see Kolmogorov and Fomin (1999)) of the payoff functional (14), we immediately obtain (26). Applying the main lemma of calculus of variations (see, e.g., Kolmogorov and Fomin (1999)) yields (27).

The condition (26) states that in expectation, the difference between the informed trader's marginal revenue (given by v) and marginal cost (given by $\overline{P} - X_N^* \overline{P}'$) is equal to zero for *all* trading strategy variations $\delta X(\cdot)$. The first-order condition (27) states that since (26) is satisfied for all variations $\delta X(\cdot)$, the difference between the marginal revenue and marginal cost is equal to zero point-by-point.

The marginal cost component includes both the average price the informed trader expects to pay, $\overline{P}$, and a price impact term, $X_N^* \overline{P}'$, since he takes into account his monopoly power. This first-order condition is different from the linear case in that the the slope of the expected supply function, $\overline{P}'(X_N^*(v), X_N^*(\cdot))$, is a potentially non-constant random variable depending on v, reflecting the possibility that $\overline{P}(x, X_N(\cdot))$ is a potentially non-linear function of x. In the linear case, the slope of the supply schedule, λ, is a constant.

Proposition 1 states the first-order condition for the BNE as unconditional expectation which must be equal to zero for all admissible trading strategy variations $\delta X(\cdot)$. It also replaces conditions which hold for all trading strategy variations $\delta X(\cdot)$ with conditions which hold for all realizations of the liquidation value v when an equilibrium trading strategy is chosen.

Next, making use of the results of Lemma 1, we can put a linear bound on the pricing rule $P(y)$ on a real axis when $y \in R$.

Corollary 1. *The pricing rule $P(\cdot)$ is linearly bounded on a real axis, i.e. there exist two real constants a and b such that*

$$|P(y)| \leq a|y| + b, \quad y \in R, \quad a \in R, \quad b \in R. \tag{28}$$

Proof: See Appendix.

Discussion of Proof: In order to prove Corollary 1, we make use of the results of Lemma 1 which allow one to represent the pricing rule as a conditional expectation of the liquidation value v, combined with convexity considerations. One should note that the bound (28) only holds on a real axis $y \in R$ and therefore does not imply that the pricing rule has to be a linear function.

When the trading strategy $X(\cdot)$ is linear, the functionals $P(y; X(\cdot))$ and $\overline{P}(x; X(\cdot))$ are identical linear functions given by equation (8), since zero-mean noise term u has no effect when P is linear in y. However, the functions $P(y; X(\cdot))$ and $\overline{P}(x; X(\cdot))$ are generally different from each other and are not given by simple closed-form expressions when the trading strategy $X(\cdot)$ is non-linear. In this case, the noise term u makes $\overline{P}$ a smoothed version of P. This property enables us to put constraints on the possible functional form of the expected price functional $\overline{P}$ as a function of the informed order flow x. This technical result is summarized bellow.

Result 2. *Consider the expected price functional as a function of the informed order flow* x

$$\begin{aligned}\overline{P}(x) &= \mathrm{E}_u\left[P(x+u)\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} du \exp\left(-\frac{u^2}{2}\right)P(x+u), \quad x \in R.\end{aligned} \tag{29}$$

Then the expected price has the following properties:

1. $\overline{P}(x; X(\cdot))$ *can be analytically continued to the complex plane* $x \in \mathbb{C}$*, and the resulting complex functional is an entire function in its first argument.*

2. On a real axis, the expected price can be represented in the form

$$\overline{P}(x) = \psi(x) + \varkappa_1 x, \tag{30}$$

with $\varkappa \in R$*, and* $\psi(\cdot)$ *being a real function which is uniformly bounded along with all derivatives* $\left|\psi^{(k)}(x)\right| \le B_k$*,* $k = 1, 2, ..$*, for all* $x \in R$*. Moreover, we have* $|\psi(x)| \le B_0$ *and* $\left|\psi^{(k)}(x)\right| \to 0$*,* $k = 1, 2, ..$*, in the large* x *limit,* $|x| \to \infty$*.*

Proof: See Appendix.

Since $\overline{P}(\cdot)$ is entire, the real function $\varphi(\cdot)$ is represented by a power series converging for any $x \in R$, and is also infinitely differential on the whole real axis. The above result describes the "smoothing" effect of the expectation $\mathrm{E}_u[P(x+u)]$. Essentially, it states that if the integrand in (29) is a sufficiently smooth function with nice analytic properties, then the expectation is even smoother, i.e., its analytic properties are improved.

An entire complex function is an analytic function which is defined by a unique power series *globally* convergent for all finite complex arguments. Given a point in its domain, an analytic function, by contrast, has a *local* power series representation which converges in a neighborhood of the given point. The meromorphic function can be represented as a ratio of two entire functions, and is analytic at all points in the intersection of their domains.

Importantly, the condition (27) states that the insider's optimization problem effectively reduces to the point-by-point optimization for each realization of the fundamentals v. This leads to the monotonicity property of the insider's reaction function.

Corollary 2. *Insider's reaction functional* $X_I(\cdot\ ; X_c(\cdot))$ *is monotonically increasing in its first argument for any admissible conjecture* $X_c(\cdot) \subset F$

$$X_I(v_1; X_c(\cdot)) \geq X_I(v_2; X_c(\cdot)), \quad v_1 \geq v_2. \tag{31}$$

Proof: See Appendix.

Note that since $X_I(v\ ; X_c(\cdot))$ is monotonically increasing in v, the inverse insider's reaction $V_I(\cdot; X_c(\cdot)) \equiv X_I^{-1}(\cdot; X_c(\cdot))$ is well defined and is also monotonically increasing in its first argument.

Now we turn our attention to a SOC. Kyle (1985) proves that the linear equilibrium (11) is stable with respect to small (but possibly nonlinear) perturbations. This means that there are no nonlinear equilibria that are infinitely close to the linear one and obtained by adding some small nonlinear terms to (11). Therefore, the standard linear equilibrium provides at least a local optimum for the insider's payoffs. Essentially, the proof reduces to the observation that the expected payoff is *locally* concave around the linear equilibrium (11).

In principle, this does not yet exclude the possibility that there may exist several local optima of the insider's optimization problem, in case if there are several strategies satisfying the FOC (27). In fact, we can show that all of these potential solutions also satisfy a SOC and therefore indeed may deliver a local maximum of the insider's payoffs. Proposition 2 demonstrates that all strategies that satisfy the FOC also satisfy the strong form SOC.

Proposition 2. *The strategies satisfying the first-order condition of Proposition 1, also satisfy the strong form SOC for* (14)*, and therefore deliver the local optima to the insider's problems.*

Proof: See Appendix.

Discussion of Proof: The proof utilizes the fact that the inverse insider's reaction $V_I(\cdot; X^*(\cdot)) \equiv X_I^{-1}(\cdot; X^*(\cdot))$ is monotonically increasing and is an analytic function in its first argument. Because of these two properties, the derivative $\frac{\partial}{\partial x} V_I(x; X^*(\cdot)) \geq 0$ and can take zero values only on the measure zero subset on a real axis.

Note that the linear strategy in question is the solution from Kyle (1985). The Proposition 2 states that the standard linear solution actually corresponds to the locally optimal strategy of the informed trader, when the possible strategies are not restricted to be linear. In principle, this does not yet exclude the possibility that there may exist multiple equilibria.

However, if there is a unique solution of BNE fixed point condition (15), then the only possible equilibrium can be identified as the standard linear one. As we will see below, this is indeed the case in our model.

5 Uniqueness of Equilibrium

In order to prove the uniqueness, we have to show that only linear strategy satisfies the fixed point condition (15) which includes both the price efficiency (21) and the insider's profit maximization condition (27). In other words, the insider's profit maximization problem (27) turns into a fixed point condition when the pricing rule satisfies informational efficiency condition (21).

Since the payoff functionals are continuous and differentiable, this immediately excludes the possibility of multiple competing equilibria. This means that *if the equilibrium exists*, and there is a *unique solution* of (27) with (21), then the equilibrium can be identified with the standard linear one. Indeed, as it follows from Corollary 3, the standard linear solution satisfies the FOC (27). From Proposition 2, it follows that the linear strategy also satisfies the strong SOC. Since the two necessary optimality conditions (see, e.g., Balakrishnan, 1971) are satisfied, the standard strategy remains the only candidate for the equilibrium, provided that it exists within the strategies satisfying our Assumption 1, i.e. strategies with a finite L_2 norm[3].

We start by showing that the equilibrium exists. We first observe that it is suboptimal for the insider to trade exceedingly large amounts. The reason is that the large informed demands are easily distinguished from the liquidity demand by the market makers and therefore the price impact of such trades decreases. Since the insider's expected profits are increasing in the price sensitivity, λ, these strategies are suboptimal. In other words, if the expected insider's payoff functional has an optimum, it is achieved on some strategy with finite norm. This provides a motivation for the assumption that the admissible strategies have a finite L_2 norm. Since the strategies with finite

[3] The standard linear equilibrium exists and has a finite L_2 norm, but one needs to show that it remains finite even after the linearity assumption is relaxed.

norm lead to the finite payoffs, the implication is that the insider's expected payoff is bounded from above. This is summarized in Lemma 2.

Lemma 2. *If the insider's expected payoff has an optimum, it is achieved on the strategies with finite L_2 norm. The expected payoff is bounded from above.*

Proof: See Appendix.

The economic intuition for the above result is simple, and is completely analogous to the one from Kyle (1985). If the insider wants to trade too much, he loses, since the order flow becomes more informative, and the insider can not screen his signal from the market.

In order to study the solutions of the FOC (27), we prove Theorem 1.

Theorem 1. *The FOC* (27) *with* (21) *has unique solution, which is the standard linear one.*

Proof: Appendix A provides the main proof, while Appendix B contains the alternative proof.

Discussion of Proof: The economic intuition is analogous to that corresponding to the linear case considered in (Kyle, 1985.) If the insider trades much less than what the standard linear strategy prescribes for large mispricing, the marginal costs are much smaller than the liquidation value v, and therefore the insider has to trade more. If the insider trades more than the standard linear strategy for the large v, the marginal costs are greater than v, and that pushes the insider's strategy back to the standard linear one. This way, the standard linear strategy remains the only "stable point" in the functional space. Note that since the FOC is formulated point-by-point, the "adjustment" described above also occurs point-by-point.

The proof of the above theorem relies on the analytic properties of the price functional presented in Lemma 1, Result 1 and Result 2. These properties are therefore quite important for our analysis. Intuitively, the analyticity of the insider's equilibrium strategy implies that the knowledge of the strategy in a certain range of the "mispricing" v (for example, $|v| < \varepsilon$), defines the equilibrium strategy in the whole range $v \in (-\infty; +\infty)$. This makes sense, because at the equilibrium, the price reacts to the variation of the equilibrium strategy in the whole range of v. In other words, changing the strategy $X(v)$ in a certain range of v causes the variation of the pricing rule beyond this range, since the price is also analytic. This feature is also reflected in the equilibrium. Technically, the procedure of "reconstructing" the function on the complex plane is called an *analytic continuation* (see Knopp (1996)).

Finally, Theorem 2 summarizes our main result.

Theorem 2. *Under Assumption 1 (admissibility of trading strategies), the standard linear equilibrium strategy and corresponding linear pricing rule deliver a unique BNE in the static Kyle'85 model.*

Proof: Follows immediately from Theorem 1.

6 Conclusion

We show that the standard linear insider's strategy, and correspondingly, the linear pricing rule deliver a unique equilibrium in a standard single period Kyle (1985) model. We assume that the possible insider's trading strategies are extended to arbitrary nonlinear piece-wise continuously differentiable functions of the fundamental with a finite L_2 norm. This means that there is a unique equilibrium in Kyle (1985), achieved on the standard linear insider's strategy, and the linear pricing rule.

Since this may potentially add a new dimension to modeling the informed trading, the uniqueness of equilibrium in Kyle (1985) is an important issue which requires a detailed analysis. This is the main purpose of this paper.

APPENDIX A

Proof of Lemma 1. 1. The joint distribution of v and y, is given by

$$f(y,v) = f_0(v)f(y|v) = \frac{1}{2\pi}\exp\left[-\frac{(y-X(v))^2}{2}\right]\exp\left[-\frac{v^2}{2}\right], \tag{A1}$$

where $f_0(v) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{v^2}{2}\right]$ is the "prior" distribution of the fundamental. Using the Bayes formula, we obtain from (A1)

$$f(v;X(\cdot)|X(v)+u=y) = \frac{1}{f_Y(y;X(\cdot))}\exp\left[-\frac{(y-X(v))^2}{2}\right]\exp\left[-\frac{v^2}{2}\right], \tag{A2}$$

with the marginal distribution density function

$$f_Y(y;X(\cdot)) = \int_{-\infty}^{+\infty} dv' \exp\left[-\frac{(y-X(v'))^2}{2}\right]\exp\left[-\frac{(v')^2}{2}\right]. \tag{A3}$$

Note that the p.d.f. of the conditional distribution (A2) depends on both order flow y, and the functional form of the insider's strategy $X(\cdot)$. With the notation

$$g_Y(y;X(\cdot)) = \int_{-\infty}^{+\infty} dv'v' \exp\left[-\frac{(y-X(v'))^2}{2}\right]\exp\left[-\frac{(v')^2}{2}\right], \tag{A4}$$

and p.d.f. (A2), the regret-free price (12) is given by

$$P(y;X(\cdot)) = \frac{g_Y(y;X(\cdot))}{f_Y(y;X(\cdot))}. \tag{A5}$$

Now we analyze the pricing rule (A5). Consider the following generating functional

$$Z(y,z;X(\cdot)) = \int_{-\infty}^{+\infty} dv \exp\left[-\frac{X^2(v)}{2} + yX(v) - \frac{v^2}{2} + zv\right], \tag{A6}$$

where both $y = y_1 + iy_2 \in \mathbb{C}$ and $z = z_1 + iz_2 \in \mathbb{C}$ are complex variables with $\{y_1, y_2\}$ and $\{z_1, z_2\}$ being their real and imaginary parts, respectively. In what follows, we analyze $Z(y,z;X(\cdot))$ as a function of the first two variables and therefore will use a short-hand notations $Z(y,z)$ and $P(y)$ for the generating functional and pricing rule, respectively.

From (A5), we obtain a relation between the pricing rule and the generating functional

$$P(y) = \frac{1}{Z(y,0)}\left(\frac{\partial Z(y,z)}{\partial z}\right)_{z=0} = \left(\frac{\partial}{\partial z}\right)_{z=0} \ln Z(y,z). \tag{A7}$$

For the future analysis, it is useful to introduce an expected strategy functional $Q(y)$ which is "dual" to the pricing rule $P(y)$ as

$$Q(y) = \frac{1}{Z(y,0)}\left(\frac{\partial Z(y,z)}{\partial y}\right)_{z=0} = \left(\frac{\partial}{\partial y}\right)_{z=0} \ln Z(y,z). \tag{A8}$$

We prove the following auxiliary result first.

Result 1.1 *The generating functional* $Z(y,z)$ *defined by* (A6) *is an entire function in both* y *and* z *with the orders* $1 < \rho(Z,y) \leq 2$, $1 < \rho(Z,z) \leq 2$,*and finite types.*

Proof. The proof is based on the analytic bound on the modulus $|Z(y,z)|$ in any finite region of the complex planes $y \in \mathbb{C}$ and $z \in \mathbb{C}$. Then $Z(y,z)$ is entire based on Riemann theorem[4], and the same bound enables us to evaluate the order and type of $Z(y,z)$ w.r.t. each of the complex variables y and z.

Before we proceed, we need to prove a useful property of the moment generating function which can be viewed as an analogue of the "ridge" property in the case when one of the arguments y or z belong to the real axis. From (A6), we obtain

$$\begin{aligned} |Z(y,z)| &\leq \int_{-\infty}^{+\infty} dv \left| \exp\left[-\frac{X^2(v)}{2} + yX(v) - \frac{v^2}{2} + zv\right]\right| \\ &\leq \int_{-\infty}^{+\infty} dv \exp\left[-\frac{X^2(v)}{2} + y_1 X(v) - \frac{v^2}{2} + z_1 v\right] \\ &= Z(y_1, z_1). \end{aligned} \tag{A9}$$

The above inequality means that $|Z(y,z)|$ has a "ridge" along the real axis of both y and z, in a sense that $|Z(y,z)|$ takes its maximal values when both y and z belong to the real axis

$$|Z(y_1 + iy_2, z_1 + iz_2)| \leq Z(y_1, z_1). \tag{A10}$$

Combining (A6) and (A10), we obtain

$$\begin{aligned} |Z(y,z)| &\leq \int_{-\infty}^{+\infty} dv \exp\left[-\frac{X^2(v)}{2} + y_1 X(v) - \frac{v^2}{2} + z_1 v\right] \\ &= \int_{-\infty}^{+\infty} dv \exp\left[\frac{(y_1)^2}{2} - \frac{(X(v) - y_1)^2}{2} - \frac{v^2}{2} + z_1 v\right] \\ &\leq \exp\left[\frac{(y_1)^2}{2}\right] \int_{-\infty}^{+\infty} dv \exp\left[-\frac{v^2}{2} + z_1 v\right] \\ &= \sqrt{2\pi} \exp\left(\frac{(y_1)^2}{2} + \frac{(z_1)^2}{2}\right) \leq \sqrt{2\pi} \exp\left(\frac{|y|^2 + |z|^2}{2}\right). \end{aligned} \tag{A11}$$

[4] It states that the analytic function which is bounded in any finite region of the complex plane, is entire (Knopp, 1996).

Therefore, we have

$$|Z(y,z)| \leq \sqrt{2\pi} \exp\left(\frac{|y|^2 + |z|^2}{2}\right). \tag{A12}$$

From (A12), it follows that $|Z(y,z)|$ is bounded in any finite domains of complex plains $y \in \mathbb{C}$ and $z \in \mathbb{C}$, and therefore $Z(y,z)$ is entire in each of its two variables. In fact, it also follows that $Z(y,z)$ is entire as a function of two complex variables[5]. It also immediately follows from (A12) that $Z(y,z)$ has finite orders and types in both variables, and the orders are bounded as

$$\begin{aligned} \rho\,(Z, y) &\leq 2, \\ \rho\,(Z, z) &\leq 2. \end{aligned} \tag{A13}$$

Note that in fact the bound (A12) is even stronger than (A13). Since the entire function $Z(y,z)$ can be viewed as a moment generating function in y and z, it also follows that (see Lukacs (1979))

$$\begin{aligned} \rho\,(Z, y) &> 1, \\ \rho\,(Z, z) &> 1. \end{aligned} \tag{A14}$$

Combining (A13) and (A14), we finally obtain

$$\begin{aligned} 1 < \rho\,(Z, y) &\leq 2, \\ 1 < \rho\,(Z, z) &\leq 2. \end{aligned} \tag{A15}$$

■

Since $Z(y,z)$ is entire in both arguments, its partial derivatives w.r.t. both arguments are also entire and also have finite orders and types. Combining this with (A7), we observe that the pricing rule $P\,(\cdot)$ is meromorphic as a ratio of two entire functions (see Knopp (1996)).

2. The pricing rule can be represented in a form

$$P\,(y; X\,(\cdot)) = \frac{\sum\limits_{k=0}^{+\infty} a_k\,(X\,(\cdot))\;\, y^k}{\sum\limits_{k=0}^{+\infty} b_k\,(X\,(\cdot))\;\, y^k}, \tag{A16}$$

where both series in the numerator and denominator converge in the entire complex plane and the coefficients are being real functionals $a_k\,(X\,(\cdot))$ and $b_k\,(X\,(\cdot))$ defined by

$$\begin{aligned} a_k\,(X\,(\cdot)) &= \int\limits_{-\infty}^{+\infty} dv \frac{v\,(X\,(v))^k}{k!} \exp\left[-\frac{(X\,(v))^2}{2} - \frac{v^2}{2}\right], \\ b_k\,(X\,(\cdot)) &= \int\limits_{-\infty}^{+\infty} dv \frac{(X\,(v))^k}{k!} \exp\left[-\frac{(X\,(v))^2}{2} - \frac{v^2}{2}\right]. \end{aligned} \tag{A17}$$

[5] Follows from Hartogs theorem (Shabat, 1992) since it is entire in each of the two variables taken separately.

In fact, it is straightforward (but tedious) to show that both series have infinite convergence radii defined as the maximal moduli of complex argument $|y|$ for which the series converge. The radii of convergence for $g_Y(y; X(\cdot))$ and $f_Y(y; X(\cdot))$ can be evaluated as in Rudin (1964)

$$R_a = \lim_{k\to\infty} \frac{1}{|a_k|^{1/k}} = +\infty, \qquad \text{(A18)}$$
$$R_b = \lim_{k\to\infty} \frac{1}{|b_k|^{1/k}} = +\infty,$$

but these calculations are redundant since as we know both $g_Y(y; X(\cdot))$ and $f_Y(y; X(\cdot))$ are entire by the Riemann theorem.

Example. The price $P(y; X(\cdot))$ is analytic, but may not be an entire function in y, because the entire function $f_Y(y; X(\cdot))$ in the denominator of (A16), may have zeroes on the complex plane (clearly, it has no zeroes on the real axis). Consider a step-wise strategy function

$$X_s(v) = \begin{cases} 1, & v > 0, \\ 0, & v = 0, \\ -1, & v < 0. \end{cases} \qquad \text{(A19)}$$

Substituting $X_s(v)$ into (A17), and collecting the coefficients, we obtain

$$P(y; X_s(\cdot)) = \sqrt{\frac{2}{\pi}} \tanh\left(\frac{y}{2}\right) = \sqrt{\frac{2}{\pi}} \frac{\sinh\left(\frac{y}{2}\right)}{\cosh\left(\frac{y}{2}\right)}, \qquad \text{(A20)}$$

which is an analytic and odd function of y. Clearly, the function (A20) has no poles on the real axis. However, there are poles on the imaginary axis, where the denominator has zeroes at

$$y = 2\pi\left(\frac{1}{2} + n\right), \qquad n = 0, \pm1, \pm2, ..., \qquad \text{(A21)}$$

leading to the finite convergence radius for the series obtained from (A16), as

$$R_s = \pi. \qquad \text{(A22)}$$

Note that since the function (A16) has only isolated poles, it can be extended to the whole complex plane by means of the analytic continuation as in Knopp (1996). As we have mentioned before, this is true in the general case, since (A3) has no zeroes on the real axis.

From (A16), it follows that $P(y; X(\cdot))$ is analytic in y. For this reason, we have within the convergence radius

$$P(y; X(\cdot)) = \sum_{k=0}^{+\infty} C_k(X(\cdot))\ y^k, \qquad \text{(A23)}$$

where the coefficients $\{C_k(X(\cdot))\}$ are real functionals.

For the linear strategies $X(v) = \beta v$, it follows from (A17) that

$$\frac{a_{2k}}{b_{2k}} \equiv \frac{\beta}{\beta^2 + 1} = \lambda, \qquad \text{(A24)}$$
$$b_{2k+1} \equiv 0, \qquad \forall k \geq 0,$$

leading to the linear pricing rule $P(y;X(\cdot)) = \lambda y$, consistent with Kyle (1985). For the sufficiently small $|x| < \varepsilon$, (A23) yields

$$P(y;X(\cdot)) = \lambda(X(\cdot))\ \ y + O\left(y^2\right), \tag{A25}$$

with

$$\lambda(X(\cdot)) = C_0(X(\cdot)) = \frac{a_0(X(\cdot))}{b_0(X(\cdot))}. \tag{A26}$$

Making use of (A17), we obtain

$$\begin{aligned}\lambda(X(\cdot)) &= \mathrm{E}_v\left[vX(v)\,|X(v)+u=0)\right] \\ &= Cov\left[vX(v)\,|X(v)+u=0)\right],\end{aligned} \tag{A27}$$

implying that $\lambda(X(\cdot))$ is an OLS coefficient for the nonlinear strategy $X(\cdot)$, defined with the conditional distribution (A2). For the linear strategies $X(v) = \beta v$, (A27) reduces to $\lambda = \frac{\beta}{\beta^2+1}$.

■

Proof of Result 1.

From (A7), it follows that the pricing rule is a logarithmic derivative of the generating functional $Z(y,z)$, which is an entire function with the order $\rho \leq 2$ and a finite type. Considering $Z(y,z)$ as a function of y and combining a result Hadamard theorem (Levin, 1996)[6] with (A7), we obtain

$$P(y) = \left(\frac{\partial}{\partial z}\right)_{z=0}\left\{\sum_{k=0}^{+\infty}\ln G\left(\frac{y}{y_k(z)},2\right) + D_0(z) + D_1(z)\,y + D_2(z)\,y^2\right\}, \tag{A28}$$

where, according to the Hadamard theorem, the sum on the r.h.s. is uniformly converging, $\{D_n(\cdot)\}$, $n = 0,1,2$ are smooth functions and real functionals of the trading strategy $X(\cdot)$, $\{y_k(z),\quad k = 0,1,2,..\}$ are the roots of $Z(y,z)$, and

$$\ln G\left(\frac{y}{y_k(z)},2\right) = \ln\left(1-\frac{y}{y_k}\right) + \frac{y}{y_k} + \frac{1}{2}\left(\frac{y}{y_k}\right)^2. \tag{A29}$$

We also have to take into account that since $Z(y_k(z),z) = 0$, we consider an infinitesimal shift of z to obtain

$$Z_y(y_k(z),z)\,dy_k + Z_z(y_k(z),z)\,dz = 0, \tag{A30}$$

which means that

$$\left(\frac{dy_k(z)}{dz}\right)_{z=0} = -\left(\frac{Z_z(y_k(z),z)}{Z_y(y_k(z),z)}\right)_{z=0} = -\frac{P(y_k)}{Q(y_k)}. \tag{A31}$$

Performing differentiation in (A28) and taking (A31) into account, we obtain, $n = 0,1,2$ are real functionals of the trading strategy $X(\cdot)$ and

$$P(y) = \sum_{k=0}^{+\infty} h_k(y) + C_0 + C_1 y + C_2 y^2, \tag{A32}$$

[6] The Hadamard theorem states that if an entire function f has a finite order ρ , m-th order zero at $z = 0$ and the zeroes $\{a_n, n = 1,2..\}$, it can be represented in the form of uniformly converging infinite product $f(z) = z^m \exp(P_q(z)) \prod_{n=1}^{+\infty} G\left(\frac{z}{a_n},p\right)$, where $q \leq \rho$, $p \leq \rho$, and the Weierstrass factor G defined by $G(u,p) = \left\{ {}^{1-u,\ p=0}_{(1-u)\exp\left(u+\frac{u^2}{2}+..+\frac{u^p}{p}\right),\ p>0} \right.$. In our case, $\rho = 2$ and we can take $q = p = 2$. Also, since in our case Z has no zeroes at $y = 0$, we take $m = 0$.

where $\left\{C_n = \left(\frac{d}{dz}\right)_{z=0} D_n(z)\right\}$, $n = 0, 1, 2$ are real functionals of the trading strategy $X(\cdot)$ and

$$\begin{aligned} h_k(y) &= \frac{A_k}{y_k}\frac{y}{y_k}\left(-\frac{1}{1-\frac{y}{y_k}} + 1 + \frac{y}{y_k}\right) \\ &= \frac{A_k}{y_k}\left(-\frac{1}{1-\frac{y}{y_k}} + 1 + \frac{y}{y_k} + \left(\frac{y}{y_k}\right)^2\right) \\ &= \frac{A_k}{y-y_k}\left(\frac{y}{y_k}\right)^3, \end{aligned} \tag{A33}$$

and

$$A_k = \left(\frac{Z_z(y_k(z), z)}{Z_y(y_k(z), z)}\right)_{z=0} = \frac{P(y_k)}{Q(y_k)}, \quad k = 1, 2, ... \tag{A34}$$

being the residues of the meromorphic function $P(y)$ at the simple poles $\{y_k, \quad k = 1, 2, ..\}$. Note that since P has simple poles[7], $(Z_y(y_k(z), z))_{z=0} \neq 0$, and therefore all the residues A_k are finite. This also means that although both P and Q have poles at $\{y_k, \quad k = 1, 2, ..\}$, the fractions $\frac{P(y_k)}{Q(y_k)} = A_k$ remain finite.

■

Proof of Corollary 1. Suppose that the prior distribution is $f_0(v) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{v^2}{2}\right]$ and introduce the notation

$$f_{v|y}(v, y; X(\cdot)) = f(v; X(\cdot)\,|X(v) + u = y) = \frac{\exp\left[-\frac{(y-X(v))^2}{2}\right] f_0(v)}{f_Y(y; X(\cdot))}. \tag{A35}$$

Then we have, according to Lemma 1

$$P(y; X(\cdot)) = \int dv f_{v|y}(v, y; X(\cdot)) = \mathrm{E}_{v|y}[v], \tag{A36}$$

where $\mathrm{E}_{v|y}[\cdot]$ denotes expectation with respect to the conditional distribution with the p.d.f. (A35). In what follows, we will drop the functional arguments of the distributions and pricing rule in order to simplify the notations.

Using the notation

$$f_{y|v}(y, v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - X(v))^2}{2}\right), \tag{A37}$$

the posterior distribution (A35) can be expressed as

$$f_y(v) = \frac{f_{y|v}(y, v)\, f_0(v)}{f_Y(y)}. \tag{A38}$$

[7] We have assumed that the singular part of $P(y)$ has isolated first-order poles. The case of higher-order poles is a particular case when some poles in the sum coincide (Knopp, 1996).

We have

$$\mathrm{E}_{v|y}\left[\ln\left(f_0(v)\right)\right] = \frac{1}{f_Y(y)}\int_{-\infty}^{+\infty} dv f_0(v)\, f_{y|v}(y,v)\ln\left(f_0(v)\right) \tag{A39}$$
$$= \frac{Z_1(y)}{f_Y(y)}\int_{-\infty}^{+\infty} dv \phi_y(v)\, f_0(v)\ln\left(f_0(v)\right)$$
$$= \frac{Z_1(y)}{f_Y(y)}\mathrm{E}_v^{\phi}\left[f_0(v)\ln\left(f_0(v)\right)\right]$$
$$\geq \frac{Z_1(y)}{f_Y(y)}\mathrm{E}_v^{\phi}\left[f_0(v)\right]\ln\left(\mathrm{E}_v^{\phi}\left[f_0(v)\right]\right),$$

where the p.d.f. $\phi_y(v)$ is given by

$$\phi_y(v) = \frac{1}{Z_1(y)} f_{y|v}(y,v), \tag{A40}$$

with

$$Z_1(y) = \int_{-\infty}^{+\infty} dv f_{y|v}(y,v) = \int_{-\infty}^{+\infty}\frac{dv}{\sqrt{2\pi}}\exp\left(-\frac{\left(y-X(v)\right)^2}{2}\right). \tag{A41}$$

Returning to (A39), we observe that

$$\mathrm{E}_v^{\phi}\left[f(v)\right] = \frac{f_Y(y)}{Z_1(y)}, \tag{A42}$$

and therefore (A39) yields

$$\mathrm{E}_{v|y}\left[\ln\left(f(v)\right)\right] \geq \ln\left(\frac{f_Y(y)}{Z_1(y)}\right). \tag{A43}$$

In order to make it useful, we need to estimate the *lower* bound of $\ln f_Y(y)$ and the *upper* bound of $\ln Z_1(y)$ on the r.h.s. of the above inequality. For the lower bound on $\ln f_Y(y)$, we have

$$\ln\left(f_Y(y)\right) = \ln\left(\int_{-\infty}^{+\infty} dv f_0(v)\exp\left(-\frac{\left(y-X(v)\right)^2}{2}\right)\right) \tag{A44}$$
$$= \ln\left(\mathrm{E}_v\left[\exp\left(-\frac{\left(y-X(v)\right)^2}{2}\right)\right]\right)$$
$$\geq -\mathrm{E}_v\left[\frac{\left(y-X(v)\right)^2}{2}\right],$$

where all expectations are taken with respect to the prior distribution which does not depend on y. An upper bound on $\ln Z_1(y)$ can be estimated as follows.

From (A41), we obtain

$$Z_1(y) = Z_2\exp\left(-\frac{y^2}{2}\right)\mathrm{E}_v^{\psi}\left[\exp\left(yX(v)\right)\right], \tag{A45}$$

where the expectation on the r.h.s. is evaluated with a p.d.f.

$$\psi_y(v) = \frac{1}{Z_2}\exp\left(-\frac{X^2(v)}{2}\right), \tag{A46}$$

with

$$Z_2 = \int_{-\infty}^{+\infty} \frac{dv}{\sqrt{2\pi}}\exp\left(-\frac{X^2(v)}{2}\right). \tag{A47}$$

Now, the function

$$\begin{aligned} F(y) &= \int_{-\infty}^{+\infty} dv \exp\left(-\frac{X^2(v)}{2} + yX(v)\right) \\ &= \int_{-\infty}^{+\infty} dx V'(x)\exp\left(-\frac{x^2}{2} + yx\right), \end{aligned} \tag{A48}$$

is proportional to the moment generating function for the p.d.f. $\chi(x) = V'(x)\exp\left(-\frac{x^2}{2}\right)$ This moment generating function exists and is an *entire* function for complex $y \in \mathbb{C}$ provided that $F(y)$ is finite for real y (see Lukacs, 1979, chap 7.1)[8]. If $V'(x)$ is exponentially bounded, it follows that the entire function $F(y)$ is of the order two, and its type is $b = 1/2$. In this case, we have

$$\ln F(y) \leq \frac{1}{2}y^2, \quad y \to \infty. \tag{A49}$$

Combining (A49) and (A45), we obtain for sufficiently large y

$$\ln Z_1(y) \leq c(y), \quad c(y) = o\left(y^2\right), \quad y \to \infty. \tag{A50}$$

Substituting (A50) and (A44) into (A43), we finally obtain

$$\mathrm{E}_{v|y}\left[v^2\right] \leq \mathrm{E}_v\left[(y - X(v))^2\right], \quad y \to \infty. \tag{A51}$$

Importantly, the r.h.s. is a quadratic polynomial in y, since the prior distribution does not depend on y. Therefore, the l.h.s. has to be bounded by a quadratic polynomial in y for large y. In particular, since

$$P(y) = \mathrm{E}_{v|y}[v], \tag{A52}$$

it follows that

$$(P(y))^2 = \left(\mathrm{E}_{v|y}[v]\right)^2 \leq \mathrm{E}_{v|y}\left[v^2\right], \tag{A53}$$

and therefore (A51) yields

$$(P(y))^2 \leq \mathrm{E}_v\left[(y - X(v))^2\right], \quad y \to \infty. \tag{A54}$$

[8] One sufficient condition for this is that $V'(x)$ is exponentially bounded on a real axis, i.e. there exists a finite constant γ s.t. $V'(x) \leq \exp(\gamma|x|)$ when $|x| \to \infty$. Economically, this condition means that we consider trading strategies when the insider does not trade "too little". Intuitively, this makes sense since we expect that both trading too little and too much is suboptimal for the insider. In both limits, the equilibrium inverse market depth and therefore insider's payoffs are low.

From (A54), we immediately derive a bound for the pricing rule in the form

$$\frac{|P(y)|}{|y|} \leq 1, \qquad |y| \to \infty. \tag{A55}$$

thus completing the proof.

■

Proof of Result 2. 1. According to the Result 1, the pricing rule $P(y)$ can be decomposed into the uniformly converging series (A32). Evaluating the expectation (29) and taking into account the result of Corollary 1, we obtain

$$\begin{aligned} \overline{P}(x) &= \overline{\mathrm{h}}(x) + \varkappa_0 + \varkappa_1 x, \qquad (A56) \\ \overline{\mathrm{h}}(x) &= \mathrm{E}_u \left[\sum_{k=0}^{+\infty} \frac{A_k}{x+u-y_k} \left(\frac{x+u}{y_k} \right)^3 \right], \end{aligned}$$

where the regular part is a linear function according to Corollary 1 and the poles of a singular part come as complex-conjugate pairs. Now we will show that the expectation (A56) transforms the singular part $h(y)$ of $P(y)$ into the function $\overline{\mathrm{h}}(x)$ which can be analytically continued to the entire complex plane and the resulting function is entire. First, we assume that $x \in R$ and apply the Fourier transformation method to evaluate $\overline{\mathrm{h}}(x)$. This is a legitimate operation since the series under expectation in (A56) is uniformly converging. We have

$$\begin{aligned} \overline{\mathrm{h}}(x) &= \mathrm{E}_u \left[h(x+u) \right], \qquad (A57) \\ h(y) &= y^3 \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \exp\left[iq(x+u)\right] H(q), \end{aligned}$$

and

$$H(q) = \sum_{k=0}^{+\infty} \frac{A_k}{(y_k)^3} \int_{-\infty}^{+\infty} dy \exp(-iqy) \frac{1}{y-y_k},$$

Evaluating the expectation in takes (A57) to the form

$$\begin{aligned} \overline{\mathrm{h}}(x) &= \int_{-\infty}^{+\infty} \frac{dq}{2\pi} H(q) \, \mathrm{E}_u \left[y^3 \exp(iqy) \right] \\ &= \int_{-\infty}^{+\infty} \frac{dq}{2\pi} H(q) \left(\frac{1}{i} \frac{\partial}{\partial q} \right)^3 \left(\exp(iqx) \exp\left(-\frac{q^2}{2} \right) \right) \qquad (A58) \\ &= \left(x + \frac{\partial}{\partial x} \right)^3 \int_{-\infty}^{+\infty} \frac{dq}{2\pi} H(q) \exp(iqx) \exp\left(-\frac{q^2}{2} \right). \end{aligned}$$

We adopt the following notation for the complex poles poles of the pricing rule: $\{y_k = y_k^1 + iy_k^2\}$, where the superscript distinguishes the real and imaginary parts of each pole. In what follows, we make use of the fact that the poles $\{y_k = y_k^1 + iy_k^2\}$ come in complex-conjugate pairs, and

$y_k^2 \neq 0, \forall k \geq 0$. Therefore, we adopt a convention that $y_k^2 > 0$, and all the poles can be separated as $y_k = \{y_n^+, y_n^-\} = \{y_n^1 + iy_n^2, y_n^1 - iy_n^2\}$. We also denote the residues at the poles in the upper half of the complex plane as $A_n^+ = A_n$. Then the residues in the lower half of the complex plane are $A_n^- = A_n^*$. Applying the Cauchy residue method (Knopp, 1996) yields

$$\begin{aligned} H(q) &= \sum_{k=0}^{+\infty} \frac{A_k}{(y_k)^3} \int_{-\infty}^{+\infty} dy \exp(-iqy) \frac{1}{y - y_k} \\ &= 2\pi i \sum_{n=0}^{+\infty} \begin{cases} -\frac{A_n^*}{(y_n^-)^3} \exp(-iqy_n^-), & q > 0, \\ \frac{A_n}{(y_n^+)^3} \exp(-iqy_n^+), & q < 0. \end{cases} \end{aligned} \tag{A59}$$

Combining (A59) and (A58) yields

$$\begin{aligned} \overline{\mathrm{h}}(x) = \varphi(x) = i\left(x + \frac{\partial}{\partial x}\right)^3 \sum_{n=0}^{+\infty} \int_0^{+\infty} dq \exp\left(-\frac{1}{2}q^2\right) \exp\left(-qy_n^2\right) \times \\ \times \left(\frac{A_n}{(y_n^+)^3} \exp\left[-iq\left(x - y_n^1\right)\right] - \frac{A_n^*}{(y_n^-)^3} \exp\left[iq\left(x - y_n^1\right)\right]\right), \end{aligned} \tag{A60}$$

or

$$\varphi(x) = 2\left(x + \frac{\partial}{\partial x}\right)^3 \sum_{n=0}^{+\infty} \frac{|A_n|}{|y_n^+|^3} \int_0^{+\infty} dq \sin\left[q\left(x - y_n^1\right) + \eta_n\right] \exp\left(-\frac{1}{2}q^2\right) \exp\left(-qy_n^2\right), \tag{A61}$$

where $\eta_n = \theta_n - 3\chi_n$ with $A_n = |A_n| \exp(i\theta_n)$ and $y_n^+ = |y_n^+| \exp(i\chi_n)$.

Analyzing (A60), we observe that $\varphi(x)$ and therefore $\overline{\mathrm{h}}(x)$ can be analytically continued to the entire complex plane $x \in \mathbb{C}$. Indeed, assuming that $x = x_1 + ix_2$, we obtain

$$\begin{aligned} |\varphi(x)| &\leq 2\sum_{n=0}^{+\infty} |A_n| \int_0^{+\infty} dq \exp\left(-\frac{1}{2}q^2\right) \exp\left(-qy_n^2\right) \exp(qx_2) \\ &\leq 2\sqrt{2\pi} \sum_{n=0}^{+\infty} |A_n| \exp\left(\frac{1}{2}\left(x_2 - y_n^2\right)^2\right), \end{aligned} \tag{A62}$$

which is finite in any finite domain of the complex plane, and therefore the analytic continuation of $\varphi(x)$ is entire by the Riemann theorem. Since a finite degree polynomial is a zero order entire function, it also follows that the analytic continuation of $\overline{\mathrm{h}}(x)$ is entire with a finite order $\rho \leq 2$ and type $\sigma \leq \frac{1}{2}$.

2. For the large values of x, (A61) yields

$$\varphi(x) = \sqrt{2\pi} \sum_{n=0}^{+\infty} \mathrm{Re}(A_n) \exp\left(-\frac{1}{2}\left(x - y_n^1\right)^2\right). \tag{A63}$$

Note that according to Corollary 1, the pricing rule is linearly bounded, and therefore (A63) should be also linearly bounded.

The sum (A63) is a smooth and bounded on a real axis for any finite number of terms. The question is what happens in the limit of infinite number of terms, in particular when there are "too many" poles $\{y_n\}$ located far away from the origin. This issue is resolved since the poles $\{y_n\}$ are actually the roots of the function $Z(y,z)$, which is an entire function of the order two and finite type. Therefore, we can use an upper bound for the counting function $n(r)$ of the poles (see Levin, 1996) given by $n(r) \leq Cr^{2}$[9], with a positive real constant $C \in R$.

This gives an estimate

$$|\varphi(x)| \leq \sqrt{2\pi} \int_0^{+\infty} dn(r)\, A_n(r) \exp\left(-\frac{1}{2}(x-r)^2\right) \tag{A64}$$
$$\leq \sqrt{2\pi} C \int_0^{+\infty} drr A_n(r) \exp\left(-\frac{1}{2}(x-r)^2\right) \leq \varkappa r, \quad r \to \infty,$$

which means that the density of the residues $A_n(r)$ should be asymptotically slow varying function, $A_n(r) \sim A = const$ in the large r limit. In this limit, we obtain a linear bound

$$|\varphi(x)| \leq \sqrt{2\pi} CAx = \varkappa x, \quad x \to \infty, \tag{A65}$$

and therefore we can find a non-negative constant $\chi \geq 0$ such that the function $\psi(x) = \varphi(x) - \chi x$ is smooth and bounded on a real axis $x \in R$.

Evaluating the derivatives, we obtain

$$\left|\varphi^{(k)}(x)\right| \leq \sqrt{2\pi} AC \int_0^{+\infty} drr \left(\frac{\partial}{\partial x}\right)^k \exp\left(-\frac{1}{2}(x-r)^2\right) \tag{A66}$$
$$= \sqrt{2\pi} AC \int_0^{+\infty} drr \left(-\frac{\partial}{\partial r}\right)^k \exp\left(-\frac{1}{2}(x-r)^2\right),$$

and therefore

$$\left|\varphi^{(1)}(x)\right| \leq \sqrt{2\pi} AC \Phi(x) \sim \sqrt{2\pi} B, \quad x \to \infty, \tag{A67}$$

which is consistent with (A65), and

$$\left|\varphi^{(k)}(x)\right| \leq Bx^k \exp\left(-\frac{x^2}{2}\right), \quad x \to \infty, \quad k = 2, ... \tag{A68}$$

with a finite real constant $B = AC \in R$. Note that from (A65), (A67) and (A68), it follows that the function $\psi(x)$ defined above and obtained from $\varphi(x)$ by means of "extraction" its linear component, also satisfies the conditions

$$|\psi(x)| \leq B_0, \tag{A69}$$
$$\left|\psi^{(1)}(x)\right| \sim 0, \quad x \to \infty,$$

[9] In general, an upper bound for the counting function of zeroes $n(r)$ for the entire function f of a finite order ρ given by $n(r) \leq Ar^{\rho}$ (see Levin, 1996).

and

$$\left|\psi^{(k)}(x)\right| \leq Bx^k \exp\left(-\frac{x^2}{2}\right) \to 0, \quad x \to \infty, \quad k = 2, .., \tag{A70}$$

which means that $\psi(x)$ is asymptotically a constant in the large x limit.

To summarize, we have obtained that

$$\left|\psi^{(k)}(x)\right| \to 0, \quad |x| \to \infty, \quad k = 1, 2, ..., \tag{A71}$$

which means that the function $\psi(x)$ has asymptotically small derivatives for the large values of $|x|$. This observation will be useful for the asymptotic analysis of the equilibrium below.

Example: The fact that the expected price can be analytically continued to the complex plane and the resulting function is entire, can be illustrated on the case of "step-wise" strategy considered above, leading to the pricing rule (A20). Making use of the Mittag-Leffler theorem (Knopp (1996)), we have

$$\begin{aligned} P(y; X_s(\cdot)) &= \sqrt{\frac{2}{\pi}} \tanh\left(\frac{y}{2}\right) \\ &= 4\sqrt{\frac{2}{\pi}} \sum_{k=1}^{+\infty} \frac{y}{y^2 + (2\pi)^2 \left(k - \frac{1}{2}\right)^2}, \end{aligned} \tag{A72}$$

with the Fourier transform

$$H_s(q) = 2\pi i \sqrt{\frac{2}{\pi}} Sgn(q) \sum_{k=1}^{N} \exp\left[-2\pi |q| \left(k - \frac{1}{2}\right)\right], \tag{A73}$$

where

$$Sgn(q) = \begin{cases} 1, & q > 0, \\ 0, & q = 0, \\ -1, & q < 0. \end{cases}.$$

Performing the summation in (A73) and substituting into (A58), we obtain the expected price

$$\overline{P}(x; X_s(\cdot)) = \frac{1}{\pi}\sqrt{\frac{2}{\pi}} \int_0^{+\infty} dq \frac{\sin(qx)}{\sinh(\pi q)} \exp\left(-\frac{q^2}{2}\right), \tag{A74}$$

and the estimate

$$\begin{aligned} \left|\overline{P}(x; X_s(\cdot))\right| &\leq 2\sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} dq \exp\left[q(x-\pi)\right] \exp\left(-\frac{q^2}{2}\right) \\ &= 2\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{(x-\pi)^2}{2}\right]. \end{aligned} \tag{A75}$$

Clearly, (A75) is bounded in any finite region on the complex plane, and therefore $\overline{P}(x; X_s(\cdot))$ is entire in x. As we have discussed above, the pricing rule $P(y; X(\cdot))$ in this case is a meromorphic function and it grows no faster than some exponential for large values of y. Consequently, the expected price $\overline{P}(x; X(\cdot))$ is an entire function of the finite order $\rho \leq 2$ and a finite type.

$\blacksquare$

Proof of Lemma 2. Consider the insider's strategy $X(\cdot)$ given by

$$X(v) = \alpha X_b(v), \tag{A76}$$

where $X_b(\cdot)$ is some arbitrary admissible "benchmark" strategy, and α is a real parameter. If we keep the benchmark strategy fixed, the L_2 norm of $X(\cdot)$ is proportional to α^2. In order to analyze the dependence of the expected payoff on the L_2 norm of the insider's strategy, we analyze the dependence of the insider's expected payoff on the parameter α for the arbitrary benchmark strategy $X_b(\cdot)$. As we will see below, the expected payoff monotonically increases in α for small α, and monotonically decreases when α is sufficiently large, for the arbitrary benchmark solution. By choosing some arbitrary α and $X_b(\cdot)$, the form (A76) represents a strategy with arbitrary variance.

The homothetic form does not take into account the possibility of the constant "shift" of the strategy $X_1(v) = X(v) + x_0$. Clearly, the constant shift x_0 does not change the price, and therefore the expected insider's payoffs do not depend on this shift. Since the insider is indifferent to the choice of the shift, his expected payoffs are not affected by restricting the admissible strategies to those with the finite norm.

Making use of the results of Lemma 1, we obtain

$$\overline{\Pi}(X(\cdot), Y(\cdot)) = \mathrm{E}_{v,u}\left[\frac{\partial}{\partial y} P(y; Y(\cdot))\right], \tag{A77}$$

with

$$\begin{aligned} P(y; X(\cdot)) &= \frac{\int\limits_{-\infty}^{+\infty} dv v \exp\left[-\frac{(y-X(v))^2}{2}\right]\exp\left[-\frac{v^2}{2}\right]}{\int\limits_{-\infty}^{+\infty} dv \exp\left[-\frac{(y-X(v))^2}{2}\right]\exp\left[-\frac{v^2}{2}\right]} \\ &= \frac{\int\limits_{-\infty}^{+\infty} dv v \exp\left[-\Phi(y,v)\right]}{\int\limits_{-\infty}^{+\infty} dv \exp\left[-\Phi(y,v)\right]}, \end{aligned} \tag{A78}$$

where

$$\Phi(y,v) = \frac{v^2}{2} + \frac{X^2(v)}{2} - yX(v). \tag{A79}$$

The substitution of the strategy (A76) into (A79) yields

$$\Phi(y,v,\alpha) = \frac{v^2}{2} + \frac{\alpha^2 X_b^2(v)}{2} - \alpha y X_b(v). \tag{A80}$$

In the limit of $\alpha \to \infty$, the pricing rule (A78) can be evaluated using the Laplace method (De Bruijn (1981)).

The saddle point $v_*(y)$ is obtained by maximizing the exponent $\Phi(y, v, \alpha)$ in (A80) with respect to v. Applying the FOC to $\Phi(y, v, \alpha)$

$$\left[\frac{\partial \Phi(y, v, \alpha)}{\partial v}\right]_{v=v_*(y)} = 0, \tag{A81}$$

we obtain

$$v_* + \alpha^2 X_b(v_*) X_b'(v_*) = \alpha y X_b'(v_*). \tag{A82}$$

In the limit $\alpha \to \infty$, the solution for (A82) is given by

$$X_b(v_*) = \frac{y}{\alpha}, \tag{A83}$$

and therefore

$$v_* = f\left(\frac{y}{\alpha}\right), \tag{A84}$$

where $f(\cdot)$ is an inverse function of $X_b(v_*)$ which can be in general multi-valued. Evaluating (A78) in the limit $\alpha \to \infty$, and making use of (A84), we arrive at

$$P(y; X(\cdot)) = P(y; \alpha X_b(\cdot)) = f\left(\frac{y}{\alpha}; X_b(\cdot)\right), \qquad \alpha \to \infty. \tag{A85}$$

From Lemma 1, we know that $P(y; X(\cdot))$ is analytic in y at $y = 0$. Expanding (A85), we obtain

$$P(y; \alpha X_b(\cdot)) = f(0; X_b(\cdot)) + f'(0; X_b(\cdot)) \left(\frac{y}{\alpha}\right) + O\left(\frac{y}{\alpha}\right)^2, \quad \alpha \to \infty, \tag{A86}$$

and

$$\begin{aligned} P'(y; \alpha X_b(\cdot)) &= \left(\frac{1}{\alpha}\right) f'(0; X_b(\cdot)) + O\left(\frac{1}{\alpha}\right)^2, \\ \overline{P}'(x; \alpha X_b(\cdot)) &= \left(\frac{1}{\alpha}\right) f'(0; X_b(\cdot)) + O\left(\frac{1}{\alpha}\right)^2, \qquad \alpha \to \infty. \end{aligned} \tag{A87}$$

On the other hand, it follows from Lemma 1 that

$$P(y; X(\cdot)) = P(y; \alpha X_b(\cdot)) = \lambda(\alpha X_b(\cdot))\ y + O\left(y^2\right). \tag{A88}$$

Comparing (A86) and (A88), we arrive at

$$\lambda(\alpha X_b(\cdot)) = \frac{1}{\alpha} f'(0; X_b(\cdot)), \qquad \alpha \to \infty. \tag{A89}$$

Note that the function $f(\cdot)$ in (A86) does not depend on α in the limit $\alpha \to \infty$, and the expansion (A88) is unique since $P(y; X(\cdot))$ is analytic at $y = 0$. With the notation

$$f'(0; X_b(\cdot)) = D(X_b(\cdot)), \tag{A90}$$

and making use of (A87), we obtain for the expected insider's payoff

$$\overline{\Pi}(X(\cdot), \alpha X_b(\cdot)) = \left(\frac{1}{\alpha}\right) D(X_b(\cdot)) + O\left(\frac{1}{\alpha}\right)^2, \qquad \alpha \to \infty. \tag{A91}$$

Clearly, the expected insider's payoff decreases in α asymptotically, when α is sufficiently large.

In the opposite limit of small α, $\alpha \to 0$, the pricing rule (A78) can be evaluated as

$$
\begin{aligned}
P\left(y;\alpha X_b\left(\cdot\right)\right) &\approx \frac{\int\limits_{-\infty}^{+\infty} dv v \exp\left[-\frac{v^2}{2}\right]\exp\left[\alpha y X\left(v\right)\right]}{\int\limits_{-\infty}^{+\infty} dv \exp\left[-\frac{v^2}{2}\right]\exp\left[\alpha y X\left(v\right)\right]} \\
&\approx \frac{\int\limits_{-\infty}^{+\infty} dv v \exp\left[-\frac{v^2}{2}\right]\left(1+\alpha y X\left(v\right)\right)}{\int\limits_{-\infty}^{+\infty} dv \exp\left[-\frac{v^2}{2}\right]\left(1+\alpha y X\left(v\right)\right)} \\
&= \alpha y \frac{\int\limits_{-\infty}^{+\infty} dv v X\left(v\right) \exp\left[-\frac{v^2}{2}\right]}{\int\limits_{-\infty}^{+\infty} dv \exp\left[-\frac{v^2}{2}\right]} + O\left(\alpha^2\right) \\
&= \alpha y E_v\left[v X\left(v\right)\right] + O\left(\alpha^2\right), \quad \alpha \to 0.
\end{aligned}
\tag{A92}
$$

Substituting (A92) into (A77), we obtain for the expected insider's payoff

$$
\overline{\Pi}\left(X\left(\cdot\right), \alpha X_b\left(\cdot\right)\right) = \alpha E_v\left[v X\left(v\right)\right] + O\left(\alpha^2\right), \quad \alpha \to 0. \tag{A93}
$$

The expected insider's payoff increases in α for sufficiently small α.

From (A91), it follows that the expected insider's payoff decreases in the norm of the insider's strategy, when the norm infinitely increases. This means that for some strategy with a sufficiently large norm, the insider's expected payoff increases if the norm of the strategy is reduced. The trivial exception which is not captured by (A76) is the constant shift $X_1(v) = X(v) + x_0$, where x_0 is an arbitrary constant. In this case, the insider is indifferent to the choice of the shift, and therefore his expected payoffs are not affected by restricting the admissible strategies to those with the finite norm.

On the other hand, it follows from (A93) that the expected insider's payoff increases in the norm of the insider's strategy, when the norm is sufficiently small. Combining the two results, we conclude that if the expected payoff functional achieves its maximum on some strategy, this strategy should have a finite norm. In particular, if we restrict our analysis to the L_2 functional space, the optimal strategies should have finite L_2 norm.

This leads to the following two conclusions. First, it provides a motivation for restricting the admissible strategies to those with the finite L_2 norm. Second, we conclude that the optimal insider's payoff is limited from above. Indeed, if the expected payoff achieves its maximum on some strategy with a finite norm, the upper bound is given by the expected payoff at some admissible strategy with a finite norm. This is a finite value due to (A77) and the results of Lemma 1.

■

Proof of Corollary 2. The insider's payoff (13) takes the form

$$
\Pi\left(v, x; X_c\left(\cdot\right)\right) = x\left(v - \overline{P}\left(x; X_c\left(\cdot\right)\right)\right), \tag{A94}
$$

and therefore for a given insider's strategy $X(\cdot)$ we have

$$\Pi\left(v, X\left(v\right); X_c\left(\cdot\right)\right) = X\left(v\right)\left(v - \overline{P}\left(X\left(v\right); X_c\left(\cdot\right)\right)\right). \tag{A95}$$

Now suppose that $X(\cdot)$ corresponds to the insider's reaction functional and therefore according to Proposition 1 it optimizes (A95) for each point v. Then, we have, for any two points $v_1 \geq v_2$

$$X\left(v_1\right)\left(v_1 - \overline{P}\left(X\left(v_1\right); X_c\left(\cdot\right)\right)\right) \geq X\left(v_2\right)\left(v_1 - \overline{P}\left(X\left(v_2\right); X_c\left(\cdot\right)\right)\right), \tag{A96}$$

and

$$X\left(v_2\right)\left(v_2 - \overline{P}\left(X\left(v_2\right); X_c\left(\cdot\right)\right)\right) \geq X\left(v_1\right)\left(v_2 - \overline{P}\left(X\left(v_1\right); X_c\left(\cdot\right)\right)\right). \tag{A97}$$

Subtracting the r.h.s. of (A97) from (A96) and comparing that to the difference between the r.h.s. of (A96) and the l.h.s. of (A97), we obtain

$$X\left(v_1\right)\left(v_1 - v_2\right) \geq X\left(v_2\right)\left(v_1 - v_2\right),$$

and therefore

$$\left(X\left(v_1\right) - X\left(v_2\right)\right)\left(v_1 - v_2\right) \geq 0, \tag{A98}$$

which means that $X(v_1) \geq X(v_2)$ if $v_1 \geq v_2$. Therefore, $X(\cdot)$ is monotonically increasing if it solves the insider's optimization problem. In particular, this is satisfied for the insider's reaction functional $X_I(\cdot\,; X_c(\cdot))$ for any admissible conjecture $X_c(\cdot)$.

■

Proof of Proposition 2. Evaluating the second functional variation of the payoff functional (14), with (13) it follows that

$$\delta^2\overline{\Pi}\left(X\left(\cdot\right), X_c\left(\cdot\right), \delta X\left(\cdot\right)\right) = -\mathrm{E}_v\left[\left(\delta X\left(v\right)\right)^2 R\left(X\left(v\right); X_c\left(\cdot\right)\right)\right], \tag{A99}$$

with

$$R\left(x; X_c\left(\cdot\right)\right) = 2\overline{P}'\left(x; X_c\left(\cdot\right)\right) + x\overline{P}''\left(x; X_c\left(\cdot\right)\right). \tag{A100}$$

Note that

$$\begin{aligned} R\left(x; X_c\left(\cdot\right)\right) &= \frac{\partial}{\partial x}\left\{\overline{P}\left(x; X_c\left(\cdot\right)\right) + x\overline{P}'\left(x; X_c\left(\cdot\right)\right)\right\} \\ &= \frac{\partial}{\partial x}V\left(x; X_c\left(\cdot\right)\right) \geq 0, \end{aligned} \tag{A101}$$

where the last inequality follows from Lemma 2 and the fact that the inverse of the insider's response strategy is well defined and a smooth function of its first argument. Since the inverse strategy $V'(x; X_c(\cdot))$ is also analytic in x and is not identically zero (otherwise, the inverse strategy would be a constant), it can not take zero values on any nonzero measure set on a complex plane. In particular, it can not take zero values on any nonzero measure set on a real axis.

Therefore, there exists a real number $\epsilon > 0$ such that

$$\mathrm{E}_v\left[\left(\delta X\left(v\right)\right)^2 R\left(X\left(v\right); X_c\left(\cdot\right)\right)\right] > \epsilon > 0, \tag{A102}$$

and the r.h.s. of (A99) is strictly negative

$$\delta^2\overline{\Pi}\left(X\left(\cdot\right), X_c\left(\cdot\right), \delta X\left(\cdot\right)\right) < -\epsilon < 0, \tag{A103}$$

which means that the strong form SOC is satisfied. It also follows from (A103) that the expected payoff functional is *globally concave*, and therefore the optimal insider's strategy is *unique*, for any $X_c\left(\cdot\right)$ (see Bertsekas (2003)). This is consistent with the results of Proposition 1.

In particular, at the linear strategy, (A99) reduces to

$$\delta^2\overline{\Pi}\left(X\left(\cdot\right), X_c\left(\cdot\right), \delta X\left(\cdot\right)\right) = -2\lambda\left(X_c\left(\cdot\right)\right)\mathrm{E}_v\left[\left(\delta X\left(v\right)\right)^2\right] < 0, \tag{A104}$$

where the last inequality holds since $\lambda\left(X\left(\cdot\right)\right) > 0$.

■

Proof of Theorem 1. First, combining the Result 2 and Corollary 1 yields the following representation of the expected price

$$\overline{P}\left(x\right) = \psi\left(x\right) + kx, \tag{A105}$$

where $\psi\left(x\right)$ is an entire function bounded on a real axis $x \in R$ along with all derivatives. Note that in (A105), we included a constant term into $\psi\left(\cdot\right)$, and also took into account that $\overline{P}\left(x\right)$ is linearly bounded for large values of x.

From the results of Corollary 2, it follows that the optimal trading strategies are monotonic and therefore invertible. Therefore, the results of Proposition 1 can be summarized as

$$V\left(x\right) = \frac{\partial}{\partial x}\left(x\overline{P}\left(x\right)\right). \tag{A106}$$

Combining (A106) with (A105), we obtain

$$V\left(x\right) = \left(2k + \psi'\left(x\right)\right)x + \psi\left(x\right). \tag{A107}$$

Substituting (A107) into the pricing rule (A5), we obtain

$$P\left(y; X\left(\cdot\right)\right) = \frac{\int\limits_{-\infty}^{+\infty} dx\left(2k + \frac{\partial^2}{\partial x^2}\left(x\psi\left(x\right)\right)\right)\left(2kx + \frac{\partial}{\partial x}\left(x\psi\left(x\right)\right)\right)\exp\left[-S\left(y,x\right)\right]}{\int\limits_{-\infty}^{+\infty} dx\left(2k + \frac{\partial^2}{\partial x^2}\left(x\psi\left(x\right)\right)\right)\exp\left[-S\left(y,x\right)\right]}, \tag{A108}$$

where

$$\begin{aligned} S\left(y,x\right) &= \frac{V^2\left(x\right)}{2} + \frac{x^2}{2} - yx \\ &= \frac{x^2}{2} - yx + \frac{1}{2}\left(2kx + \frac{\partial}{\partial x}\left(x\psi\left(x\right)\right)\right)^2. \end{aligned} \tag{A109}$$

Now we analyze the limit of large $y \to \infty$ making use of the asymptotic analysis methods (De Bruijn, 1981). Consider a scaling $x = \xi y$, where $\xi \in R$ remains finite when $y \to \infty$. Then (A109) yields

$$S_e(y,\xi) = y^2 F(y,\xi), \tag{A110}$$
$$F(y,\xi) = \frac{\xi^2}{2}\left(1+\left(2k+\psi'(\xi y)\right)^2\right) - \xi\left(1-\frac{1}{y}\left(2k+\psi'(\xi y)\right)\psi(\xi y)\right) + \frac{1}{2y^2}\psi^2(\xi y),$$

or

$$S_e(y,\xi) = y^2 F(y,\xi), \tag{A111}$$
$$F(y,\xi) = \frac{\left(1+(2k+\psi'(\xi y))^2\right)}{2}\left(\xi - \frac{\left(1-\frac{1}{y}(2k+\psi'(\xi y))\psi(\xi y)\right)}{1+(2k+\psi'(\xi y))^2}\right)^2 - \frac{1}{2}\frac{\left(\left(1-\frac{1}{y}(2k+\psi'(\xi y))\psi(\xi y)\right)\right)^2}{1+(2k+\psi'(\xi y))^2} + \frac{1}{2y^2}\psi^2(\xi y).$$

We will be looking for the asymptotic expansion (see De Bruijn, 1981) of the pricing rule in the limit of large y. We first evaluate the two leading-order terms of asymptotic expansion of the pricing rule P. Taking into account the property (A71), we observe that in the limit of large y, the conditional distribution characterized by the exponent (A110) is a sharp Normal and centered at the value $\xi = \xi^*$ which satisfies the following condition

$$\xi^* = \frac{\left(1-\frac{1}{y}(2k+\psi'(\xi^* y))\psi(\xi^* y)\right)^2}{1+(2k+\psi'(\xi^* y))^2}. \tag{A112}$$

Therefore, the asymptotic expansion of pricing rule is given by

$$P(y) = y\xi^*\left(2k+\psi'(y\xi^*)\right) + \psi(y\xi^*) + O\left(\frac{1}{y}\right), \tag{A113}$$

where the saddle point $\xi^* = \xi^*(y)$ still needs to be evaluated and expanded in the large y limit. In this limit, (A112) yields

$$\xi^* = \frac{1}{1+(2k+\psi'(\xi^* y))^2}\left(1-\frac{2}{y}\left(2k+\psi'(\xi^* y)\right)\psi(\xi^* y)\right) + O\left(\frac{1}{y^2}\right). \tag{A114}$$

Since, as it follows from (A71), both $\psi(\xi^* y)$ and $\psi'(\xi^* y)$ are small in the large y limit, the saddle point equation (A114) can be solved iteratively, which is also consistent with the requirement that $\xi^*(y)$ needs to be asymptotically expanded in the large y limit. We have

$$\xi^* = \frac{1}{1+(2k+\psi'(\xi^* y))^2} + O\left(\frac{1}{y}\right). \tag{A115}$$

Substituting (A115) back into (A114), we obtain

$$\begin{aligned}
P(y) &= yr_1 + r_0 + O\left(\frac{1}{y}\right), \qquad \text{(A116)}\\
r_1 &= \frac{(2k+\psi'(y\xi^*))}{1+(2k+\psi'(\xi^* y))^2},\\
r_0 &= \psi(y\xi^*) - 2\frac{(2k+\psi'(\xi^* y))\,\psi(\xi^* y)}{1+(2k+\psi'(\xi^* y))^2}.
\end{aligned}$$

Proceeding analogously, we obtain for the expected price

$$\begin{aligned}
\overline{P}(x) &= x\overline{r}_1 + \overline{r}_0 + O\left(\frac{1}{x}\right), \qquad \text{(A117)}\\
\overline{r}_1 &= \frac{(2k+\psi'(\xi^* x))}{1+(2k+\psi'(\xi^* x))^2},\\
\overline{r}_0 &= \psi(\xi^* x) - 2\frac{(2k+\psi'(\xi^* x))\,\psi(\xi^* x)}{1+(2k+\psi'(\xi^* x))^2}.
\end{aligned}$$

Comparing with (A105), we finally obtain

$$k = \frac{(2k+\psi'(\xi^* x))}{1+(2k+\psi'(\xi^* x))^2}, \qquad \text{(A118)}$$

and

$$\psi(x) = \psi(\xi^* x) - 2\frac{(2k+\psi'(\xi^* x))\,\psi(\xi^* x)}{1+(2k+\psi'(\xi^* x))^2}.$$

From (A118), it follows that

$$\psi'(\xi^* x) = \frac{1}{2k} - 2k \pm \sqrt{\frac{1}{4k^2} - 1}, \qquad \text{(A119)}$$

which means that $\psi'(\xi^* x)$ remains a constant in the large x limit. Since $|\psi(x)|$ is bounded on a real axis $x \in R$, (A119) can only hold if the r.h.s. is zero, i.e.

$$\frac{1}{2k} - 2k \pm \sqrt{\frac{1}{4k^2} - 1}, \qquad \text{(A120)}$$

which can be only satisfied for $k = \frac{1}{2}$. Also, $\psi'(x) = 0$ and $\psi(x) = C_0 = const$. Since $|\psi(x)| \to 0$ in the large x limit, $C_0 = 0$, and we obtain a standard linear pricing rule $P(y) = \frac{1}{2}y$.

Proceeding analogously, we can obtain the higher order terms of the asymptotic expansion of the pricing rule P in the large y limit. A straightforward (but tedious) calculation shows that, analogous to what happens to the leading order asymptotic correction terms, none of the nonlinear terms in the asymptotic expansion can survive under the fixed point equilibrium requirement (A106).

Therefore, the only possibility is a linear pricing rule, $\psi(x) \equiv const$. Following the standard argument from Kyle (1985), we immediately obtain the equilibrium trading strategy as $X(v) = v$.

■

APPENDIX B

Alternative Proof of Theorem 1. We are going put an upper bound on the pricing rule $P(y)$ from (A7) by applying the *Nevanlinna's Lemma* to it. The part of the Nevanlinna's Lemma related to our problem is as follows. Consider an entire function $f(z)$. As stated in Nevanlinna (1970) (see Eq. (3.2) on p.244), we have the following bound for the Nevanlinna characteristic of the logarithmic derivative $\frac{f'(z)}{f(z)}$ of the function $f(z)$ when the complex variable z belongs to a circle $|z| = \rho$

$$m\left(\rho, \frac{f'}{f}\right) \leq 2\ln^+\left(T\left(\rho, f\right)\right) + 2\ln^+ r + 2\ln^+\left(\frac{1}{r-\rho}\right) + 2\ln^+\left(\frac{1}{\rho}\right) + 2\ln^+\ln^+\left(\frac{1}{|f(0)|}\right) + 2, \tag{B1}$$

where $0 < \rho < r$ and the Nevanlinna measure $m(\rho, F)$ is defined by

$$m(\rho, F) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \ln^+\left|F\left(\rho e^{i\theta}\right)\right|.$$

Since the function f in our case is an entire function Z, $T(\rho, f) = m(\rho, f) = \sigma\rho^2 + const$, with σ being a positive real number.

The proof is given in Li (2011) for a more general case of several complex variables. See also the original exposition in Nevanlinna (1970).

Applying Nevanlinna's lemma to Pricing Rule. Note that the argument z in (A6) plays the role of the prior expectation (which we typically assume to be zero). However, in order to directly apply the Nevanlinna's lemma, we will assume that $|z| = \rho \neq 0$, and therefore (A7) is redefined.

Define

$$\begin{aligned} P_*(y, w) &\equiv \left(\frac{\partial}{\partial z}\right)_{z=w} \ln Z(y, z) = \frac{Z_w(y, w)}{Z(y, w)}, \\ Q_*(y, w) &\equiv \left(\frac{\partial}{\partial y}\right) \ln Z(y, w) = \frac{Z_y(y, w)}{Z(y, w)}. \end{aligned} \tag{B2}$$

Then we have

$$\begin{aligned} P(y) &= P_*(y, w = 0) = \left(\frac{\partial}{\partial z}\right)_{z=0} \ln Z(y, z), \\ Q(y) &= Q_*(y, w = 0) = \left(\frac{\partial}{\partial y}\right)_{z=0} \ln Z(y, z). \end{aligned} \tag{B3}$$

By analogy, define also

$$\begin{aligned} \overline{P}_*(x, w) &= \mathrm{E}_u\left[P_*(x + u, w)\right], \\ \overline{Q}_*(x, w) &= \mathrm{E}_u\left[Q_*(x + u, w)\right]. \end{aligned} \tag{B4}$$

Then we have

$$P'(y) = \left(\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial z}\right)_{z=0} \ln Z(y,z) = \left(\frac{\partial}{\partial z}\right)_{z=0} Q_*(y,z), \tag{B5}$$

and

$$\overline{P}'(x) = \frac{\partial}{\partial x}\overline{P}(x) = \frac{\partial}{\partial x}\mathrm{E}_u\left[P(x+u)\right] = \mathrm{E}_u\left[P'(x+u)\right], \tag{B6}$$

and therefore

$$\overline{P}'(x) = \left(\frac{\partial}{\partial w}\right)_{w=0} \overline{Q}_*(x,w). \tag{B7}$$

Now, we apply Nevanlinna's and Cauchy's lemmas to show that there exists a *regular system of contours*[10] $\{\Gamma_n, n = 1, 2, ...\}$ and a finite number $n \in N$ such that

$$|Q_*(y,w)| \leq \Pi_n^{(w)}(y), \tag{B8}$$

when $y \in \{\Gamma_n, n = 1, 2, ...\}$, where $\Pi_n^{(w)}(y)$ is a polynomial of y of degree n. The coefficients of the polynomial in the r.h.s. may depend on w.

It is straightforward to show that n is determined by the convergence exponent of the poles of Q_* which also correspond to the zeroes of $Z(y,z)$, which is an entire function of the second order and finite type. Therefore, the convergence exponent is $p = 2$ and $\Pi_n^{(w)}(y) = k(w)\,y$ is linear in y. This immediately puts bounds on both terms in the Mittag-Leffler expansion (see Knopp, 1996) of the function $Q_*(y,z)$ given by

$$Q_*(y,z) = \sum_{k=0}^{+\infty} g_k(y) + K_0(z) + K_1(z)\,y, \tag{B9}$$

where $\{K_n, n = 0, 1\}$, are real functionals of the trading strategy $X(\cdot)$ and functions of z, and

$$g_k(y) = \frac{1}{y_k}\left(-\frac{1}{1-\frac{y}{y_k}} + 1 + \frac{y}{y_k}\right) = \frac{1}{y - y_k}\left(\frac{y}{y_k}\right)^2. \tag{B10}$$

Making use of the relation (B5) and integrating w.r.t. y, we obtain the expansion for the pricing rule (A32) obtained in the Result 1 by means of using the Weierstrass expansion for the generating functional $Z(y,z)$. This alternative derivation can be viewed as a "robustness check" for the Result 1.

■

[10] Consider a system of closed contours $\{\Gamma_n\}, n = 1, 2, ...,$ and denote by $\widehat{\Gamma}_n$ the set of points on a complex plane located inside of the contour Γ_n. The regular system of contours is such that 1) for any n, $0 \in \widehat{\Gamma}_n$, 2) $\widehat{\Gamma}_n \in \widehat{\Gamma}_{n+1}$, and 3) $S_n/d_n \leq C$, where S_n and d_n are the length and the distance from the origin of the contour Γ_n, respectively, and $C \in R$ is a finite constant (see e.g. Shabat (1992)).

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