Uniqueness of equilibrium in static Kyle (1985) model

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Overview

- Motivation
- Related Literature
- Our Approach to Uniqueness
- Kyle (1985) Model
- Definition of Equilibrium
- Main Results
- Conclusion

Motivation: Why do we care?

- One-shot Kyle (1985) (along with its modifications) is a major "work horse" in Microstructure (both theory and empirics)
- Sets guidelines for some empirical work: *how* do the informed arbitrageurs exploit their informational advantage?
- *Or:* What are the optimal strategies of informed strategic arbitrageurs?
- Q: Are these strategies and market reaction unique?

Related Literature

1. Modification of Kyle (1985) setting: Insider observes total OF

   Rochet and Vila (1994)

   MMs' and Insider's information sets are *nested*

2. Dynamic continuous time models version of Kyle (1985)

   Back (1992)

   Price process is continuous, the situation is similar to Rochet and Vila (1994): nested info sets

3. Twicking the assumptions of Kyle (1985) – discrete strategies (*finite action space*)

   Cho and Karoui (2000)
   Result: There may be nonlinear strategies; claim uniqueness

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- **Exact same** setting and assumptions, except for allow for *nonlinear* insider's strategy and pricing rule
- Try to prove uniqueness in this case; turns out to be more difficult!

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- In what follows, we normalize $\sigma_0 = \sigma_u = 1$

Nash Equilibrium in Kyle (1985) Model

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  - **Market efficiency:** market makers' expected profits equal to zero, conditional on observing the order flow, and taking the informed trader's trading strategy as given
  $$P^*(y) = \mathrm{E}[v|X^*(v) + u = y]$$

  Result: *There exists a unique equilibrium in which $X^*(\cdot)$ and $P^*(\cdot)$ are linear functions*
  $$X^*(v) = v, \qquad P^*(y) = \frac{1}{2}y.$$

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- Stick to the above notation

Bayesian Nash Equilibrium (BNE)

- I observes realization v and trades x, MMs' conjecture $X_M(\cdot)$; I's expected payoff

$$\Pi(v, x; X_M(\cdot)) = \mathrm{E}_u[(v - P(x + u; X_M(\cdot)))x]$$

Definition

The Bayesian Nash Equilibrium (BNE thereafter) strategy, $X_N^*(\cdot)$, is defined by the fixed-point condition

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- **Interpretation:** I's *optimal response* same as MMs' *conjecture*

Admissible Strategies

Definition

A trading strategy $X(\cdot)$ *is admissible iff* $\mathrm{E}_v[(X(v))^2] < \infty$

Assumption *Insider's optimal strategies are admissible*

- In what follows, we show that the strategies with infinitely increasing L_2 norm are suboptimal

Pricing Rule: Analytic Properties

Lemma

Let f_Y be marginal probability density of y given $X(\cdot)$

$$f_Y(y; X(\cdot)) = \int_{-\infty}^{+\infty} dv f_{V,Y}(v, y; X(\cdot)).$$

Let g_Y denote expectation over v

$$g_Y(y; X(\cdot)) = \int_{-\infty}^{+\infty} dv v f_{V,Y}(v, y; X(\cdot))$$

Then the pricing rule P is given by

$$P(y; X(\cdot)) = \frac{g_Y(y; X(\cdot))}{f_Y(y; X(\cdot))}$$

For a given trading strategy $X(\cdot)$, $P(y)$ is meromorphic in order flow y

Representation of Pricing Rule

Corollary

Taking as given a trading strategy $X(\cdot)$, *the pricing rule* $P(y)$ *can be expressed as uniformly converging series*

$$P(y) = \sum_{k=0}^{+\infty} h_k(y) + C_0 + C_1 y + C_2 y^2,$$

where $\{C_n\}$, $n = 0, 1, 2$ *are real functionals of the trading strategy* $X(\cdot)$ *and*

$$h_k(y) = \frac{A_k}{y - y_k}\left(\frac{y}{y_k}\right)^3,$$

with $\{A_k\}$ *and* $\{y_k\}$, $k = 0, 1, 2, ..$*being complex valued functionals of* $X(\cdot)$ *representing the residues and poles of the meromorphic function* P, *respectively*

Expected Price Functional

Definition

Let $\overline{P}(x; X(\cdot))$ denote the expected price obtained by the informed trader when he trades quantity x and the market makers believe he is using the trading strategy $X(\cdot)$. Then $\overline{P}(x; X(\cdot))$ is a functional defined by

$$\overline{P}(x; X(\cdot)) = \mathrm{E}_u \left[P(x + u; X(\cdot)) \right]$$

- Note: In linear case, $\overline{P}$ is identical to P

First Order Condition (FOC)

Corollary

A necessary condition for a BNE is that for all admissible variations $\delta X(\cdot)$ *we have*

$$
\begin{aligned}
0 &= \delta_1 \overline{\Pi}\left(X_N^*(\cdot), \delta X(\cdot); X_N^*(\cdot)\right) \\
&= \mathrm{E}_v\left[\left\{v - \overline{P}\left(X_N^*(v); X_N^*(\cdot)\right) - X_N^*(v)\,\overline{P}'\left(X_N^*(v); X_N^*(\cdot)\right)\right\}\delta X(v)\right],
\end{aligned}
$$

and therefore

$$
\overline{P}\left(X_N^*(v); X_N^*(\cdot)\right) - X_N^*(v)\,\overline{P}'\left(X_N^*(v); X_N^*(\cdot)\right) = v
$$

- **Interpretation:** Marginal profit = Marginal cost

Functional Derivative

Technical tool

Definition

The functional differential of the price with respect to the strategy $X(\cdot)$ is

$$\delta_X \overline{P}\left(x; X(\cdot), \delta X(\cdot)\right) = \lim_{\varepsilon \to 0} \left\{ \frac{\overline{P}\left(x; X(\cdot) + \varepsilon \delta X(\cdot)\right) - \overline{P}\left(x; X(\cdot)\right)}{\varepsilon} \right\},$$

provided that the limit exists for every $\delta X(\cdot)$ (from the same functional space), and defines a functional, linear and bounded in $\delta X(\cdot)$

- Useful tool for our analysis
- Can be viewed as an extension of the *directional derivative* of functions depending on several variables

Second Order Condition (SOC)

Corollary

Insider's reaction functional $X_I(\cdot\,; X_c(\cdot))$ *is monotonically increasing in its first argument for any admissible conjecture* $X_c(\cdot) \subset F$

$$X_I(v_1; X_c(\cdot)) \geq X_I(v_2; X_c(\cdot)), \quad v_1 \geq v_2$$

- Combining this with FOC, we obtain

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Corollary

The strategies satisfying the first-order condition of Proposition 1, also satisfy the strong form SOC, and therefore deliver the local optima to the insider's problem

Pricing Rule: Linear Bound

Corollary

The pricing rule $P(\cdot)$ is linearly bounded on a real axis, i.e. there exist two real constants a and b such that

$$|P(y)| \leq a|y| + b, \quad y \in R, \quad a \in R, \quad b \in R$$

- Note: this does not imply that P is linearly bounded in a complex plane

Main Results

Lemma

If the insider's expected payoff has an optimum, it is achieved on the strategies with finite L_2 norm. The expected payoff is bounded from above

- The FOC is a *necessary* condition

Theorem

The FOC has unique solution, which is the standard linear one

Theorem

Under Assumption 1 (admissibility of trading strategies), the standard linear equilibrium strategy and corresponding linear pricing rule deliver a unique BNE in the static Kyle'85 model

Example: Linear Case

- Insider's strategy: $X\left(v\right) = \beta v$

$P(y) = \lambda y$ with $\lambda = \frac{\beta}{1+\beta^2}$

- **Insider's expected payoff:** $\overline{\pi}_I = E\left[\beta v\left(v - \lambda y\right)\right] = \lambda = \frac{\beta}{1+\beta^2}$
- **Non-monotonic w.r.t.** β (suboptimal to trade infinite amount)

Example: Linear Case

- Insider's strategy: $X(v) = \beta v$
- Pricing rule (reaction to the OF $y = X(v) + u$):

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- **Insider's expected payoff:** $\overline{\pi}_I = E\left[\beta v\left(v - \lambda y\right)\right] = \lambda = \frac{\beta}{1+\beta^2}$
- **Non-monotonic w.r.t.** β (suboptimal to trade infinite amount)

Make use of

$$\overline{P}(x) = \psi(x) + kx,$$

where $\psi(x)$ is an entire function bounded on a real axis $x \in R$ along with all derivatives

Inverse optimal strategy

$$V(x) = \frac{\partial}{\partial x}\left(x\overline{P}(x)\right)$$

Combining, we obtain

$$V(x) = \left(2k + \psi'(x)\right)x + \psi(x)$$

Proof (contd)

Pricing rule

$$P\left(y;X\left(\cdot\right)\right)=\frac{\int\limits_{-\infty}^{+\infty}dx\left(2k+\frac{\partial^2}{\partial x^2}\left(x\psi\left(x\right)\right)\right)\left(2kx+\frac{\partial}{\partial x}\left(x\psi\left(x\right)\right)\right)\exp\left[-S\right]}{\int\limits_{-\infty}^{+\infty}dx\left(2k+\frac{\partial^2}{\partial x^2}\left(x\psi\left(x\right)\right)\right)\exp\left[-S\right]}$$

with

$$\begin{aligned}S\left(y,x\right)&=\frac{V^2\left(x\right)}{2}+\frac{x^2}{2}-yx\\&=\frac{x^2}{2}-yx+\frac{1}{2}\left(2kx+\frac{\partial}{\partial x}\left(x\psi\left(x\right)\right)\right)^2\end{aligned}$$

- Analyze the limit of large $y\to\infty$ making use of the asymptotic analysis methods

Proof (contd)

Pricing rule

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with

$$\begin{aligned} S\left(y, x\right) &= \frac{V^2\left(x\right)}{2} + \frac{x^2}{2} - yx \\ &= \frac{x^2}{2} - yx + \frac{1}{2}\left(2kx + \frac{\partial}{\partial x}\left(x\psi\left(x\right)\right)\right)^2 \end{aligned}$$

- Analyze the limit of large $y \to \infty$ making use of the asymptotic analysis methods
- Result: $\psi\left(x\right) = 0, \ k = 1/2$

Conclusion

- We establish a uniqueness in the standard single period Kyle (1985) model
- FOC is a necessary condition
- FOC (along with efficiency) can only be satisfied on standard linear strategies