Short-Term Momentum and Long-Term Reversal in General Equilibrium*

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Abstract

We evaluate the ability of the Lucas [25] tree and the Alvarez-Jermann [3] models, both with homogeneous as well as heterogeneous beliefs, to generate a time series of excess returns that displays both short-term momentum and long-term reversal, i.e., positive autocorrelation in the short-run and negative autocorrelation in the long-run. Our analysis is based on a methodological contribution that consists in (i) a recursive characterisation of the set of constrained Pareto optimal allocations in economies with limited enforceability and belief heterogeneity and (ii) an alternative decentralisation of these allocations as competitive equilibria with endogenous borrowing constraints. We calibrate the model to U.S. data as in Alvarez and Jermann [4]. The sign of the autocorrelations predicted by the Lucas tree model or the Alvarez-Jermann model with correct beliefs coincide with the data for some preferences parameters. However, we find that only the Alvarez-Jermann model with heterogeneous beliefs delivers autocorrelations that not only have the correct sign but are also of magnitude similar to the US data when the preferences parameters are disciplined to match both the average annual risk-free rate and equity premium.

Keywords: Heterogeneous beliefs, Endogenously Incomplete Markets, Financial Markets Anomalies, Limited Enforceability, Constrained Pareto Optimality, Recursive Methods

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1 Introduction

Over the last several years, a large volume of empirical work has documented that excess returns in the stock market appear to exhibit *short-term momentum*, that is positive autocorrelation, in the short to medium run and *long-term reversal*, that is negative autocorrelation, in the long run (see Moskowitz et al [26], Poterba and Summers [29] and Lo and MacKinlay [24]).

There is a tendency to interpret these properties of excess returns as a rejection of standard models of asset pricing and so they are known as *financial markets anomalies*.[1] Although this interpretation might be correct, it is not apparent why standard models of asset pricing cannot generate the pattern of autocorrelations found in the data.

In this paper we evaluate the ability of two standard general equilibrium asset pricing models to generate a time series of excess returns that display both short-term momentum and long-term reversal. We consider both an economy without frictions, the Lucas [25] tree model adapted to allow for stochastic growth as in Mehra and Prescott [27], as well as an economy where credit frictions arise due to limited enforceability, the Alvarez-Jermann [3] model. For each of them we analyse both the case of homogeneous and heterogeneous beliefs. We say that a model's predictions are *qualitatively* accurate if the sign of the predicted autocorrelations coincide with that of their empirical counterparts for some preferences parameters. We say that a model's predictions are *quantitatively* accurate if its predicted autocorrelations are both of the same sign and order of magnitude as in the data when one sets the discount rate and coefficient of risk aversion to match the average annual risk-free rate of 0.8% and equity premium of 6.18%. We calibrate the stochastic process of individual income and aggregate growth rates of a two-agent economy to aggregate and US household data as in Alvarez and Jermann [4]. For each case, we first ask whether its predictions are qualitatively accurate. If the answer is positive, we study whether the model's predictions are quantitatively accurate as well.

The autocorrelation of excess returns is zero when the expectations are computed using the so-called equivalent martingale measure or, as we call it, the market belief. Yet, as it has been noticed long time ago, the *empirical* excess returns could be autocorrelated.[2] This is because (i) the empirical autocorrelations converge to the autocorrelations computed with respect to the true probability measure and (ii) the market belief, typically, differs from the true probability measure.[3] Loosely speaking, short-term momentum and long-term reversal occurs if the conditional equity premium is pro-cyclical in the short-run but counter-cyclical in the long-run.

We consider a pure exchange economy where the state of nature follows a finite first-order time homogeneous Markov process. There is a finite number of infinitely-lived agents who are subjective utility maximisers and have heterogeneous beliefs regarding the transition probability matrix.[4]

[1] Fama and French [18] suggest this interpretation as a logical possibility, while Poterba and Summers [29] argue that these properties of excess returns should be attributed to "price fads."

[2] See Leroy [23] or Lucas [25], for example.

[3] Note that (ii) is true even if some agents have correct beliefs because the market belief adjusts the true probability to take into account the effect of time and risk on the marginal valuation of future consumption.

[4] Our framework is general enough to accommodate bounded or unbounded aggregate growth and priors with and

We first consider the Lucas tree model and its competitive equilibria (CE), i.e. full risk sharing equilibria. Under some mild assumptions, CE prices and excess returns converge to those of an economy where only agents with correct beliefs have positive wealth (see Sandroni [31] and Blume and Easley [10]). Thus, we restrict attention to the case where everybody has correct beliefs.[5] We find that the predictions of the Lucas model are qualitatively accurate, its failure is only of a quantitative nature. Indeed, the autocorrelations are of an order of magnitude smaller than in the data.

Next we consider competitive equilibria with solvency constraints (CESC) that prevent agents to attain full risk sharing, an equilibrium concept very close to the one used in Alvarez and Jermann [3]. Following Alvarez and Jermann [3] and Kehoe and Levine [20], we say that an allocation is enforceable if agents would at no time be better off reverting permanently to autarky. We say that an allocation is constrained Pareto optimal (CPO) if it is optimal within the set of enforceable allocations. Our analysis of CESC allocations is based on a methodological contribution that complements the techniques developed by Spear and Srivastava [32] and Abreu et al [2]. Indeed, we first provide a complete recursive characterisation of the set of constrained Pareto optimal allocations and a version of the principle of optimality for these economies. We also show how to decentralise a CPO allocation as a CESC using a suitable adaptation of the methodology that Beker and Espino [5] develop to decentralise a Pareto optimal allocation of an economy with belief heterogeneity as a CE.

When both agents have homogeneous beliefs, i.e. the Alvarez and Jermann [3] model, the predictions are qualitatively accurate. However, they are not quantitative accurate and the failure is even starker than that of the Lucas model since not even the signs are correct. The different quantitative predictions of the Lucas and Alvarez and Jermann models arise because the calibrated labor income shocks display counter-cyclical cross-sectional variance, as Krueger and Lustig [22] point out. In particular, this counter-cyclical property of the labor income shocks makes the conditional equity premium counter-cyclical in the short-run as in Chien and Lustig [16].

To assess the impact of belief heterogeneity on CESC, we assume agent 1 has correct beliefs and agent 2 has dogmatic beliefs that are pessimistic about the persistency of the expansion state and correct otherwise. The presence of solvency constraints ensures that the consumption of every agent is bounded away from zero, i.e. both agents survive. We set the beliefs of the pessimistic agent so that the time series of returns matches the historical short-term momentum and we found that the model does a very good job explaining long-term reversal as well.

The main lesson is that if one insists in that some agents must eventually have correct beliefs, then perpetual pessimism, belief heterogeneity and limited enforceability are three ingredients that together give a quantitative explanation for short-term momentum and long-term reversal in a general equilibrium setting. Pessimism makes the market more pessimistic at expansions than at recessions which makes the conditional equity premium pro-cyclical in the short-run. This is the main driving force to explain both short-term momentum and long-term reversal. Belief heterogeneity and limited enforceability make the welfare weights change as time and uncertainty unfold and so it increases

without the true transition matrix in its support. If some agent has the truth in his prior's support but others do not, belief heterogeneity does not vanish.

[5]Note that since we are interested in asymptotic results, this restriction is without loss in generality

the volatility of the stochastic discount factor. This excess volatility does not vanish because limited enforceability makes the pessimistic agent survive. Consequently, the average equity premium can be matched for levels of risk aversion more moderate than those that would be necessary in an otherwise identical economy without limited enforceability or belief heterogeneity.

We are not the first to use pessimism to explain asset pricing puzzles in general equilibrium. However, most of the previous papers are representative agent models. Abel [1], assumes the belief of the representative agent is characterised by pessimism and doubt and he shows that these effects reduce the risk-free rate and increase the equity premium. Cogley and Sargent [13] focus on the quantitative effects of pessimism on the equity premium. However, neither of these authors tackle the effect of pessimism on the autocorrelations of excess returns. Cecchetti et al [12] explain several anomalies, including long-term reversal, but they are silent about short-term momentum. Our approach differs from theirs in one important aspect. In their model the stochastic discount factor has a non stationary behaviour because they assume not only that the representative agent is pessimist but also that she believes the endowment growth follows a peculiar non-stationary process. In our model, instead, the agents correctly believe the true process is stationary while the non stationary behaviour of the stochastic discount factor arises endogenously due to changes in the wealth distribution.

Cogley and Sargent [14] combine both pessimism and belief heterogeneity but they focus only on their effect on the market price of risk on a finite sample. Although the pessimistic agent ends up learning, they show that, for a plausible calibration of their model, it takes a long time for the effect of large pessimism on CE asset prices to be erased unless the agents with correct beliefs own a large fraction of the initial wealth.

Finally, Cao [11] and Cogley et al [15] also combine the same three ingredients to study the dynamics of asset prices. Cao focuses on survival and excess volatility of asset prices. Cogley et al focus on the wealth dynamics of a bond economy when solvency constraints are exogenously given and proportional to the agents' income.

This paper is organised as follows. Section 2, describes the model. Our methodological contribution is introduced in sections 3 and 4. Section 5 provides a statistical and economic characterisation of short-term momentum and long-term reversal. In section 6, we evaluate the ability of CE and CESC allocations to generate short-term momentum and long-term reversal. Section 7 provides a final discussion. Proofs are gathered in the Appendix.

2 The Model

We consider a one-good infinite horizon pure exchange stochastic economy. In this section we establish the basic notation and describe the main assumptions.

2.1 The Environment

Time is discrete and indexed by $t = 0, 1, 2, ...$ The set of possible states of nature is $S \equiv \{1, ..., K\}$. The state of nature at date zero is known and denoted by $s_0 \in S$. The set of partial histories up to date

$t \geq 1$, S^t, is the $t-$Cartesian product of S with typical element $s^t = (s_1, ..., s_t)$. S^∞ is the set of infinite sequences of the states of nature and $s = (s_1, s_2, \cdots)$, called a path, is a typical element. For every partial history s^t, $t \geq 1$, a *cylinder* with base on s^t is the set $C(s^t) \equiv \{\tilde{s} \in S^\infty : \tilde{s} = (s^t, \tilde{s}_{t+1}, \cdots)\}$ of all paths whose t initial elements coincide with s^t. Let $\mathcal{F}_t$ be the σ-algebra that consists of all finite unions of the sets $C(s^t)$. The σ-algebras $\mathcal{F}_t$ define a filtration $\mathcal{F}_0 \subset ... \subset \mathcal{F}_t \subset ... \subset \mathcal{F}$ where $\mathcal{F}_0 \equiv \{\emptyset, S^\infty\}$ is the trivial $\sigma-$algebra and $\mathcal{F}$ is the σ-algebra generated by the algebra $\bigcup_{t=1}^{\infty} \mathcal{F}_t$.

Let Δ^{K-1} be the $K-1$ dimensional unit simplex in $\Re^K$. We say that $\pi : S \times S \to [0,1]$ is a transition probability matrix if $\pi(\cdot\,|\xi) \in \Delta^{K-1}$ for all $\xi \in S$. If $\{s_t\}$ follows a first-order time-homogeneous Markov process with a $K \times K$ transition probability matrix π, then P^π denotes the probability measure on $(S^\infty, \mathcal{F})$ uniquely induced by π. Let Π^K denote the set of $K \times K$ transition probability matrices and Π^K_{++} be the subset consisting of all $K \times K$ transitions probability matrices with strictly positive entries. $\mathcal{B}\left(\Pi^K\right)$ are the corresponding Borel sets and $\mathcal{P}(\Pi^K)$ is the set of probability measures on $\left(\Pi^K, \mathcal{B}\left(\Pi^K\right)\right)$. The following assumption will be used for the characterisation of the dynamics in Sections 5-6 where we need to be explicit about the true data generating process (henceforth, dgp).

A.0 The true dgp is given by P^{π^*} for some $\pi^* \in \Pi^K_{++}$.

Definition. *A state of nature ξ is strongly persistent if $\pi^*\left(\xi|\,\xi\right) \geq \psi^*\left(\xi\right)$, where ψ^* is the invariant distribution associated with π^**

2.2 The Economy

There is a single perishable consumption good every period. The economy is populated by I (types of) infinitely-lived agents where $i \in \mathcal{I} = \{1, ..., I\}$ denotes an agent's name. A consumption plan is a sequence $\{c_t\}_{t=0}^{\infty}$ such that $c_0 \in \mathbb{R}_+$ and $c_t : S^\infty \to \mathbb{R}_+$ is $\mathcal{F}_t-$measurable for all $t \geq 1$ and $\sup_{(t,s)} c_t(s) < \infty$. Given s_0, the agent's consumption set, $\mathbb{C}(s_0)$, is the set of all consumption plans.

2.2.1 Beliefs

P_i is the probability measure on $(S^\infty, \mathcal{F})$ that represents agent i's prior. Throughout this paper, we assume that each agent i assigns positive probability to every partial history s^t, i.e., $P_i(C(s^t)) > 0$ for all s^t. We say that agent i believes the dgp consists of draws from a (fixed) transition probability matrix if for every event $A \in \mathcal{F}$

$$P_i\left(A\right) = \int_{\Pi^K} P^\pi\left(A\right) \mu_{i,0}\left(d\pi\right),$$

where $\mu_{i,0} \in \mathcal{P}(\Pi^K)$ is agent i's *belief* over the unknown transition probability matrix. Let $\mu_0 \equiv (\mu_{1,0}, ..., \mu_{I,0})$ denote the collection of beliefs of the agents at date zero.

A1 Agent i believes the true dgp consists of draws from a transition probability matrix and either
a. $\mu_{i,0}$ has countable support.
b. $\mu_{i,0}$ has density $f_{i,0}$ with respect to Lebesgue that is continuous.

Assumption A1 implies that posterior beliefs depend on the history only through the prior. Indeed, Bayes' rule implies that beliefs evolve according to

$$\mu_{i,s^{t+1}}(D) = \frac{\int_D \pi(s_{t+1}\,|s_t)\;\mu_{i,s^t}(d\pi)}{\int_{\Pi^K} \pi(s_{t+1}\,|s_t)\;\mu_{i,s^t}(d\pi)} \quad \text{for any } D \in \mathcal{B}(\Pi^K), \tag{1}$$

where $\mu_{i,s^0} = \mu_{i,0}$ is given at date 0.

The following assumptions when coupled with $A1$, impose more structure on the agent's prior.[6]

A2 Agent i has the true transition probability matrix in the support of her prior. That is, either
a. $\mu_{i,0}(\pi^*) > 0$ if $\mu_{i,0}$ has countable support.
b. $f_{i,0}(\pi^*) > 0$ if $\mu_{i,0}$ has density $f_{i,0}$ with respect to Lebesgue.

We say that agent i is *dogmatic* if his belief is a point mass probability measure on some $\pi_i \in \Pi^K$, i.e., $\mu_i^{\pi_i} : \mathcal{B}\left(\Pi^K\right) \to [0,1]$ is given by

$$\mu_i^{\pi_i}(B) \equiv \begin{cases} 1 & \text{if } \pi_i \in B \\ 0 & \text{otherwise.} \end{cases}$$

Agents with dogmatic beliefs satisfy A.1 but they satisfy A.2 only if $\pi_i = \pi^*$. Let $\pi = (\pi_1, ..., \pi_I)$ and $\mu^\pi \equiv (\mu_1^{\pi_1}, ..., \mu_I^{\pi_I})$. The following assumption defines a large class of heterogeneous dogmatic beliefs that we use In Proposition 4.

A3 There exists $\xi^* \in S$ such that $\frac{\pi_1(\xi^*|\xi^{**})}{\pi_2(\xi^*|\xi^{**})}\frac{\pi_1(\xi^{**}|\xi^*)}{\pi_2(\xi^{**}|\xi^*)} \neq 1$ for some $\xi^{**} \in S$.

2.2.2 Preferences

Agents' preferences have a subjective expected utility representation that is time separable, i.e., for every $c_i \in \mathbb{C}(s_0)$ her preferences are represented by

$$U_i^{P_i}(c_i) = E^{P_i}\left(\sum_{t=0}^{\infty} \rho_{i,t}\; u_i(c_{i,t})\right),$$

where $u_i : \mathbb{R}_+ \to \{-\infty\} \cup \mathbb{R}$ is continuously differentiable, strictly increasing, strictly concave and $\lim_{x\to 0} \frac{\partial u_i(x)}{\partial x} = +\infty$ and $\rho_{i,t}$ is agent i's multi-period stochastic discount factor recursively defined by

$$\rho_{i,t+1}(s) = \beta(s_t, \mu_{i,s^t})\rho_{i,t}(s) \text{ for all } t \text{ and } s,$$

where $\rho_{i,0}(s_0) \in (0,1)$ is given and $\beta(\xi, \cdot) : \mathcal{P}(\Pi^K) \to (0,1)$ is continuous for all ξ and uniformly bounded above by $\overline{\beta} \in (0,1)$.[7,8] If agent i has dogmatic beliefs, we write $\beta_i(\xi) \equiv \beta(\xi, \mu_i^{\pi_i})$ for all ξ.

2.3 Feasibility, Enforceability and Constrained Optimality

Agent i's endowment at date t is a time-homogeneous function of the current state of nature that we denote by $y_i(\xi) > 0$ for all ξ. The aggregate endowment is denoted by $y(\xi) \equiv \sum_{i=1}^I y_i(\xi) \leq \overline{y} < \infty$. Let $y_{i,t}(s) \equiv y_i(s_t)$ and $y_t(s) \equiv y(s_t)$.

[6] We adopt the convention of writing $\mu_i(\{\pi^*\})$ as $\mu_i(\pi^*)$.
[7] We allow for utility functions unbounded from below.
[8] In the standard case where $\beta(\xi,\mu) = \beta$ for all ξ, $\rho_{i,t}(s) = \beta^t$ for all $t \geq 1$ and s.

Given a consumption plan $c_i \in \mathbb{C}(s_0)$, define

$$U_i(c_i)(s^t) = u_i(c_i(s^t)) + \beta(s_t, \mu_{i,s^t}) \sum_{\xi'} \pi_{\mu_{i,s^t}}(\xi' \,|s_t) U_i(c_i)(s^t, \xi') \text{ for all } t \text{ and } s^t,$$

where $\pi_{\mu_{i,s^t}}(\xi' \,|s_t) = \int \pi(\xi' \,|s_t)\, \mu_{i,(s^t,\xi')}\,(d\pi)$ where $\mu_{i,(s^t,\xi')}$ is obtained from μ_{i,s^t} using (1). When c_i is the endowment of agent i, we simply write $U_i(s_t, \mu_{i,s^t})$ to make clear that the utility attained from consuming the individual endowment forever can be expressed as a function only of s_t and μ_{i,s^t}.

Let $Y\,(s_0)$ be the set of feasible allocations. Given (s_0, μ_0), a feasible allocation $\{c_i\}_{i=1}^I$ is *enforceable* if $U_i(c_i)(s^t) \geq U_i(s_t, \mu_{i,s^t})$ for all t, s^t and i. Let $Y^E(s_0, \mu_0) \subset Y\,(s_0)$ be the set of enforceable allocations. A feasible allocation $\{c_i\}_{i=1}^I$ is *Pareto optimal (PO)* if there is no alternative feasible allocation $\{\tilde{c}_i\}_{i=1}^I \in Y\,(s_0)$ such that $U_i^{P_i}(\tilde{c}_i) > U_i^{P_i}(c_i)$ for all i. An enforceable allocation $\{c_i\}_{i=1}^I$ is *constrained Pareto optimal (CPO) given* (s_0, μ_0) if there is no other enforceable allocation $\{\tilde{c}_i\}_{i=1}^I \in Y^E\,(s_0, \mu_0)$ such that $U_i^{P_i}(\tilde{c}_i) > U_i^{P_i}(c_i)$ for all i.

Given (s_0, μ_0), define the *utility possibility correspondence* by

$$\mathcal{U}(s_0, \mu_0) = \{\tilde{u} \in \mathbb{R}^I : \exists\ \{c_i\}_{i=1}^I \in Y\,(s_0)\,,\ \tilde{u}_i \leq U_i^{P_i}(c_i) \quad \forall i\},$$

and the *enforceable utility possibility correspondence* by

$$\mathcal{U}^E(s_0, \mu_0) = \{\tilde{u} \in \mathbb{R}^I : \exists\ \{c_i\}_{i=1}^I \in Y^E(s_0, \mu_0),\ U_i(s_0, \mu_0) \leq \tilde{u}_i \leq U_i^{P_i}(c_i) \quad \forall i\}.$$

Given (s_0, μ_0), the set of CPO allocations can be characterised as the solution to the following planner's problem with welfare weights $\alpha \in \mathbb{R}_+^I$:

$$v^*(s_0, \alpha, \mu_0) \equiv \sup_{\{c_i\}_{i=1}^I \in Y^E(s_0,\mu_0)} \quad \sum_{i=1}^I \alpha_i\ E^{P_i} \left(\sum_t \rho_{i,t}\ u_i(c_{i,t}) \right). \tag{2}$$

It is straightforward to prove that (2) can be rewritten as

$$v^*(s_0, \alpha, \mu_0) = \sup_{\tilde{u}\, \in\, \mathcal{U}^E(s_0,\mu_0)} \quad \sum_{i=1}^I \alpha_i\ \tilde{u}_i. \tag{3}$$

The maximum is attained since the objective function is continuous and the constraint set is compact.

2.3.1 An Economy with Aggregate Growth

Let $g : S \to \mathbb{R}_+$ and $\epsilon_i : S \to (0,1)$ denote the (stochastic) growth rate and income share of agent i, respectively. Then,

$$y_t(s) = g(s_t)\ y_{t-1}(s) \textit{ and } y_{i,t}(s) = \epsilon_i(s_t)\ y_t(s) \textit{ for all } i,\ t \textit{ and } s. \tag{4}$$

Definition. *An economy where the aggregate endowment satisfies (4), the discount factor is non-stochastic and preferences display constant relative risk aversion is called a growth economy. A baseline growth economy is a growth economy where $I = 2$, $K = 4$, $g\,(1) = g\,(3)$ and $g\,(2) = g\,(4)$.*

Our specification of the discount factor let us accommodate growth as in Alvarez and Jermann [3]. Indeed, we now argue that the set of enforceable allocations of a growth economy can be characterised by studying the set of enforceable allocations of an economy with constant aggregate endowment and an stochastic discount factor.

Let $\widehat{c}_{i,t}(s) = c_{i,t}(s)/y_t(s)$, $\widehat{y}_{i,t}(s) = y_{i,t}(s)/y_t(s) = \epsilon_i(s_t)$ for all i, s and t. Notice that $\widehat{y}_t(s) = \sum_{i=1}^{I} \widehat{y}_{i,t}(s) = 1$ for all s and t. Then,

$$\widehat{U}_i(\widehat{c}_i)(s^t) = u_i(\widehat{c}_{i,t}(s)) + \widehat{\beta}(s_t, \mu_{i,s^t}) \sum_{\xi'} \widehat{\pi}_{\mu_{i,s^t}}(\xi' \,|s_t) \widehat{U}_i(\widehat{c}_i)(s^t, \xi') \text{ for all } t \text{ and } s,$$

where

$$\widehat{\pi}_{\mu_{i,s^t}}(\xi' \,|s_t) = \frac{\pi_{\mu_{i,s^t}}(\xi' \,|s_t) g(\xi')^{1-\sigma}}{\sum_{\tilde{\xi}} \pi_{\mu_{i,s^t}}(\tilde{\xi} \,|s_t) g(\tilde{\xi})^{1-\sigma}} \quad \text{and} \quad \widehat{\beta}(s_t, \mu_{i,s^t}) = \beta \sum_{\xi'} \pi_{\mu_{i,s^t}}(\xi' \,|s_t) g(\xi')^{1-\sigma}.$$

As in Mehra and Prescott [27], expected utility is well defined if

$$\sup_{\xi,\mu_i} \left\{ \beta \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi) g(\xi')^{1-\sigma} \right\} < 1. \tag{5}$$

Let $\widehat{c}_i \equiv \{\widehat{c}_{i,t}\}_{t=0}^{\infty}$ for all i and $\widehat{y} \equiv \{\widehat{y}_t\}_{t=0}^{\infty}$. We define the *normalised stationary economy* associated to the growth economy by $\left(\widehat{y}, \left\{\widehat{c}_i, \widehat{U}_i\right\}_{i\in\mathcal{I}}\right)$.

Finally, $\{c_i\}_{i=1}^{I}$ is an enforceable allocation in the growth economy iff $\{\widehat{c}_i\}_{i=1}^{I}$ is an enforceable allocation in the normalised stationary economy. Also, the preference orderings are identical in the two corresponding economies and the discount factor is stochastic if and only if the growth rate is.

3 A Recursive Approach to CPO

In this section, we provide the recursive characterisation of the set of CPO allocations and a version of the Principle of Optimality for economies with heterogeneous beliefs and limited enforceability.

3.1 The Recursive Planner's Problem

In Appendix A we show that $v^* : S \times \mathbb{R}_+^I \times \mathcal{P}(\Pi) \to \mathbb{R}$ solves the functional equation[9]

$$v^*(\xi, \alpha, \mu) = \max_{(c, w'(\xi'))} \sum_{i=1}^{I} \alpha_i \left\{ u_i(c_i) + \beta(\xi, \mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi) \; w_i'(\xi') \right\}, \tag{6}$$

subject to

$$c_i \geq 0, \qquad \sum\nolimits_{i=1}^{I} c_i = y(\xi), \tag{7}$$

$$u_i(c_i) + \beta(\xi, \mu_i) \sum\nolimits_{\xi'} \pi_{\mu_i}(\xi' \,|\xi) \; w_i'(\xi') \geq U_i(\xi, \mu_i), \tag{8}$$

$$w_i'(\xi') \geq U_i(\xi', \mu_i'(\xi, \mu)(\xi')) \qquad \text{for all } \xi', \tag{9}$$

$$\min_{\tilde{\alpha} \in \mathbb{R}_+^I} \left[v^*(\xi', \tilde{\alpha}, \mu'(\xi, \mu)(\xi')) - \sum\nolimits_{i=1}^{I} \tilde{\alpha} \; w_i'(\xi') \right] \geq 0 \text{ for all } \xi', \tag{10}$$

where

$$\mu_i'(\xi, \mu_i)(\xi')\,(D) \equiv \frac{\int_D \pi(\xi' \,|\xi) \; \mu_i\,(d\pi)}{\int \pi(\xi' \,|\xi) \; \mu_i\,(d\pi)} \text{ for any } D \in \mathcal{B}(\Pi^K),$$

$$\mu'(\xi, \mu)(\xi') \equiv (\mu_1'(\xi, \mu_1)(\xi') ... \mu_I'(\xi, \mu_I)(\xi')),$$

and $\alpha'(\xi')$ is the solution to problem (10) for state of nature ξ'.

[9] In section 3.1 we abuse notation and let c to be a non-negative vector and c_i its i^{th} component.

In the recursive dynamic program defined by (6) - (10), the current state of nature, ξ, captures the impact of changes in aggregate output while (α, μ) summarises and isolates the history dependence introduced by the $\mathcal{B}$-margin of heterogeneity, $\frac{\int \pi(\xi'|\xi)\mu_i'(\xi,\mu)(\xi')(d\pi)}{\int \pi(\xi'|\xi)\mu_j'(\xi,\mu)(\xi')(d\pi)}$, introduced by Beker and Espino [5] and limited enforceability.[10] The planner takes as given (ξ, α, μ) and allocates current consumption and continuation utility levels among agents. The optimisation problem defined in condition (10) characterises the set of continuation utility levels attainable at $(\xi', \mu'(\xi, \mu)(\xi'))$ (see Lemma A.1 in Appendix A).[11] The weights $\alpha'(\xi, \alpha, \mu)(\xi')$ that attain the minimum in (10) are the weights that support next period allocation.

Any (c, w', α') that satisfies (7) - (10) will be referred as a *set of policy functions.* Given (s_0, α_0, μ_0), we say the policy functions (c, α') generate an allocation $\{c_t\}_{t=0}^{\infty} \in \mathbb{C}(s_0)^I$ if

$$\begin{aligned} c_{i,t}(s) &= c_i(s_t, \alpha_t(s)), \\ \alpha_{t+1}(s) &= \alpha'(s_t, \alpha_t(s), \mu_{s^t})(s_{t+1}), \\ \mu_{i,s^{t+1}} &= \mu_i'(s_t, \mu_{i,s^t})(s_{t+1}), \end{aligned}$$

for all i, $t \geq 0$ and $s \in S^{\infty}$, where $\alpha_0(s) = \alpha_0$ and $\mu_{i,s^0} = \mu_{i,0}$.

It follows by standard arguments that the corresponding optimal consumption policy function, $c_i(\xi, \alpha)$, is the unique solution to

$$c_i(\xi, \alpha) + \sum_{h \neq i} \left(\frac{\partial u_h}{\partial c_h} \right)^{-1} \left(\frac{\alpha_i}{\alpha_h} \frac{\partial u_i(c_i(\xi, \alpha))}{\partial c_i} \right) = y(\xi).$$

where $\left(\frac{\partial u_h}{\partial c_h} \right)^{-1}$ denotes the inverse of the function $\frac{\partial u_h}{\partial c_h}$.

The following Theorem states our version of the Principle of Optimality. It shows that there is a one-to-one mapping between the set of CPO allocations and the allocations generated by the optimal policy functions solving (6) - (10).

Theorem 1. *An allocation $(c_i^*)_{i=1}^I$ is CPO given (ξ, α, μ) if and only if it is generated by the set of policy functions solving (6) - (10).*

Given $\alpha_{-i} \in \mathbb{R}_+^{I-1}$, define

$$\underline{\alpha}_i(\xi, \mu)(\alpha_{-i}) = \min_{\alpha_i} \left\{ \alpha_i \in \mathbb{R}_+ : u_i(c_i(\xi, (\alpha_i, \alpha_{-i}))) + \beta(\xi, \mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \mid \xi)\ w_i'(\xi, (\alpha_i, \alpha_{-i}), \mu)(\xi') = U_i(\xi, \mu) \right\}$$

where $c_i(\xi, \alpha)$ and $w_i'(\xi, (\alpha_i, \alpha_{-i}), \mu)(\xi')$ are the maximisers in problem (6) - (10). For $I = 2$, we simply write $\underline{\alpha}_1(\xi, \mu)$ and $\underline{\alpha}_2(\xi, \mu)$.

[10] To be more precise, Beker and Espino define the $\mathcal{B}$-margin as the ratio of the priors about the states of nature in the following t periods while here it is the ratio of the priors about the realisations of next period state of nature.

[11] To understand condition (10) notice that the utility possibility correspondence is convex, compact and contains its corresponding frontier. The frontier of a convex set can always be parameterised by supporting hyperplanes. Thus, a utility level vector w is in the utility possibility correspondence if and only if for every welfare weight α the hyperplane parameterised by α and passing through w, αw, lies below the hyperplane generated by the utility levels attained by the CPO allocation corresponding to that welfare weight α, attaining the value $v(\xi, \alpha, \mu)$. This is why we must have $\alpha w \leq v(\xi, \alpha, \mu)$ for all α or, equivalently, $\min_{\tilde{\alpha}} [v(\xi, \tilde{\alpha}, \mu) - \tilde{\alpha} w] \geq 0$.

The following Proposition shows that constraint (8) can be ignored by restricting the welfare weights to lie in $\Delta(\xi,\mu) \equiv \{\alpha \in \Delta^{I-1} : \alpha_i \geq \underline{\alpha}_i(\xi,\mu)(\alpha_{-i}) \text{ for all } i\}$.[12]

Proposition 2. *Let $(\xi,\mu) \in S \times \mathcal{P}(\Pi^K)$. (i) If $\alpha \in \Delta(\xi,\mu)$, then constraint (8) does not bind at any solution to (6) - (10). (ii) If $\alpha \notin \Delta(\xi,\mu)$, then there exists some $\tilde{\alpha} \in \Delta(\xi,\mu)$ such that $c(\xi,\alpha) = c(\xi,\tilde{\alpha})$.*

The (normalised optimal) law of motion for the welfare weights, $\alpha'_{i,cpo}(\xi,\alpha,\mu)(\xi')$, follows from the first order conditions with respect to the continuation utility levels for each individual. In the two-agent case, the CPO law of motion for agent 1's welfare weight is

$$\alpha'_{1,cpo}(\xi,\alpha,\mu)(\xi') = \begin{cases} \underline{\alpha}_1(\xi',\mu) & \text{if } \alpha'_{1,po}(\xi,\alpha,\mu)(\xi') < \underline{\alpha}_1(\xi',\mu) \\ 1-\underline{\alpha}_2(\xi',\mu) & \text{if } \alpha'_{1,po}(\xi,\alpha,\mu)(\xi') > 1-\underline{\alpha}_2(\xi',\mu) \\ \alpha'_{1,po}(\xi,\alpha,\mu)(\xi') & \text{otherwise} \end{cases}$$

where

$$\alpha'_{1,po}(\xi,\alpha,\mu)(\xi') = \frac{\alpha_1 \int \pi(\xi' \,|\xi)\mu'_1(\xi,\mu)\,(\xi')\,(d\pi)}{\alpha_1 \int \pi(\xi' \,|\xi)\mu'_1(\xi,\mu)\,(\xi')\,(d\pi) + \alpha_2 \int \pi(\xi' \,|\xi)\,(\xi')\,\mu'_2(\xi,\mu)\,(d\pi)}$$

is the PO law of motion for agent 1's welfare weight that depends only on the extent to which beliefs are heterogeneous as captured by the $\mathcal{B}$-margin. The CPO law of motion for agent 1's welfare weight, instead, combines two effects: belief heterogeneity and limited enforceability. To understand the impact of each effect we discuss them in isolation. If beliefs are heterogeneous but enforceability is perfect, the CPO law of motion for agent 1's welfare weight becomes the PO law of motion. Therefore, the changes in agent 1's welfare weight are purely driven by changes in the $\mathcal{B}$-margin. If beliefs are homogeneous and enforceability is imperfect, the case analysed by Alvarez and Jermann [3], the CPO law of motion requires the next period welfare weight to be equal to the current one unless that conflicts with the need to provide incentives to avoid the agent to revert to autarky, i.e. there is some state of nature for which the current welfare weight does not lie in the interval defined by the corresponding minimum enforceable weights. Therefore, the changes in agent 1's welfare weights are purely driven by the need to satisfy enforceability. If agents have heterogeneous beliefs and enforceability is limited, both effects might interact. Consequently, changes in agent 1's welfare weight are driven by the $\mathcal{B}$-margin unless that conflicts with enforceability.

3.1.1 Discussion

There are at least two alternative approaches to solve recursively the problem defined by (3). To simplify the exposition we assume there are only two agents. The first alternative was developed by Thomas and Worral [35] and Kocherlakota [21].[13] Instead of using welfare weights to parameterised

[12] The proof of Proposition 2 is included in the supplementary material.

[13] Thomas and Worral [35] study the efficient distribution of risk between a risk-neutral firm and a risk-averse worker in a partial equilibrium setting without commitment. This simplified framework let them describe the Pareto frontier recursively. Kocherlakota [21] consider a general equilibrium setting and claims that their same technique can be applied to his problem.

the allocations, the planner chooses current feasible consumption and continuation utilities for both agents in order to maximise agent 1's utility subject to three restrictions: (i) agent 2's utility is above some pre-specified level, (ii) feasible allocations are period-by-period enforceable and (iii) continuation utility levels lie in the next period utility possibility correspondence. Very importantly, these last two conditions imply that the value function defines the constraint set. The second alternative, developed in Beker and Espino [5], studies directly the operator defined by (6) - (10).

Since both approaches use the value function to define the constraint set, it is not clear that any of the associated operators satisfies Blackwell's discounting (sufficient) condition for a contraction. When enforceability constraints are ignored, Beker and Espino [5] show that discounting is satisfied if the operator is properly restricted. With enforceability constraints, however, their approach cannot be applied. The difficulty can be explained as follows. For any function v that defines the constraint set, there might be some positive constant, $a > 0$ such that $v + a$ enlarges the feasible set of choices of continuation utilities with respect to v. Although $v + a$ is still an affine linear transformation of v, it gives some room to deal with enforceability and that conflicts with discounting. As a matter of fact, uniqueness is not satisfied since the function $f(\xi, \alpha, \mu) = \sum_{i=1}^{I} \alpha_i U_i(\xi, \mu_i)$ is also a fixed point of the operator defined by (6) - (10).

Our strategy relates to the seminal idea pioneered by Abreu, Pearce and Stacchetti [2] (discussed in Alvarez and Jermann [4] in a setting with limited enforceability). They construct an operator that iterates directly on the utility possibility correspondence and then the value function (and the corresponding optimal policy functions) are recovered from the frontier of the fixed point of that operator. Our approach follows their idea but it iterates directly on the utility possibility frontier parameterised by welfare weights. To implement this strategy, it is key that the utility possibility correspondence is convex-valued, a property that is not assumed in Abreu, Pearce and Stacchetti [2] since they are interested in non-convex problems.[14]

3.2 Computation

For many purposes it is important to have an algorithm capable of finding the value function v^*. Let $\tilde{v}$ be the value function solving the recursive problem when the enforceability constraints are ignored (see Beker and Espino [5]). Evidently, $v^*(\xi, \alpha, \mu) \leq \tilde{v}(\xi, \alpha, \mu)$ for all (ξ, α, μ).

Proposition 3. *Let $v_0 = \tilde{v}$ and $v_n = T(v_{n-1})$ for all $n \geq 1$. Then, $\{v_n\}$ is a monotone decreasing sequence and* $\lim_{n\to\infty} v_n = v^*$.

3.3 The Welfare Weights Dynamic with Dogmatic Beliefs

In this section we assume there are two agents who have dogmatic beliefs. $\Omega \equiv \{(\xi, \alpha) \in S \times \Delta^1 : \alpha \in \Delta(\xi, \mu^\pi)\}$ is the state space and $\mathcal{G}$ its $\sigma-$algebra. For $t \geq 0$, Ω^t is the $t-$cartesian product of Ω with typical element $\omega^t = (\xi_0, \alpha_0, ..., \xi_t, \alpha_t)$ and $\Omega^\infty = \Omega \times \Omega \times ...$ is the infinite product of the state space with typical element $\omega = (\omega_0, \omega_1, ...)$. $\mathcal{G}_{-1} \equiv \{\varnothing, \Omega^\infty\}$ is the trivial $\sigma-$algebra, $\mathcal{G}_t$ is the

[14]They study bang-bang solutions for problems where non-convexities arise due to incentive compatibility constraints.

$\sigma-$algebra that consists of all the cylinder sets of length t. The $\sigma-$algebras $\mathcal{G}_t$ define a filtration $\mathcal{G}_{-1} \subset \mathcal{G}_0 \subset ... \subset \mathcal{G}_t \subset ... \subset \mathcal{G}^\infty$, where $\mathcal{G}^\infty \equiv \mathcal{G} \times \mathcal{G} \times ...$ is the $\sigma-$algebra on Ω^∞.

The law of motion for the welfare weights, α'_{cpo}, coupled with π^* define a time-homogeneous transition function on the states of nature and the welfare weights, $F_{cpo} : \Omega \times \mathcal{G} \to [0,1]$, given by

$$F_{cpo}\left[(\xi, \alpha), \mathcal{S} \times \mathcal{A}\right] = \sum_{\xi' \in \mathcal{S},\ \alpha'_{cpo}(\xi,\alpha)(\xi') \in \mathcal{A}} \pi^*\left(\xi' \,|\xi\right) \text{ for all } (\mathcal{S} \times \mathcal{A}) \in \mathcal{G}$$

The transition function F_{cpo} together with a probability measure ψ on $(\Omega, \mathcal{G})$ induces a unique probability measure $P^{F_{cpo}}(\psi, \cdot)$ on $(\Omega^\infty, \mathcal{G}^\infty)$. We define the operator T^* on the space of probability measures on $(\Omega, \mathcal{G})$ as

$$T^*\psi\left(\mathcal{S}, \mathcal{A}\right) = \int F_{cpo}\left((\xi, \alpha), \mathcal{S} \times \mathcal{A}\right) d\psi \text{ for all } (\mathcal{S} \times \mathcal{A}) \in \mathcal{G}$$

We use standard arguments to show that T^* has a unique invariant measure on $(\Omega, \mathcal{G})$ and that the distribution of states converges weakly to that measure.

Proposition 4. *Suppose $I = 2$ and A.0, A1 and A3 holds for both agents. Then there exists a unique invariant measure $\psi_{cpo} : \mathcal{G} \to [0,1]$. Moreover, ψ_{cpo} is globally stable and non-degenerate.*

Actually, Beker and Espino [6] show that CPO allocations are *never* PO for a large class of heterogeneous priors in any two-agent economy. Moreover, our numerical simulations led us to conjecture that, typically, the support of the invariant distribution has a finite number of points.

4 Competitive Equilibrium with Solvency Constraints

In this section we define a competitive equilibrium with solvency constraints (CESC). In Section 4.1 we show that CPO allocations can be decentralised as CESC and study the determinants of the financial wealth distribution. In section 4.2 we study the limit distribution of wealth and consumption in a CESC.

Every period t, after observing s^t, agents trade both the consumption good and a complete set of Arrow securities in competitive markets. Security ξ' issued at date t pays one unit of consumption if next period's state of nature is ξ' and 0 otherwise. We denote by $q_t^{\xi'}(s)$ and $a_{i,t}^{\xi'}(s)$ the price of Arrow security ξ' and agent i's asset holdings, respectively, at date t on path s. Let $a_{i,-1}^{\xi'} = 0$ for all ξ', $a_{i,t} = \left(a_{i,t}^1, ..., a_{i,t}^K\right)$ and $a_i \equiv \{a_{i,t-1}\}_{t=0}^\infty$ for all i. Prices are in units of the date–t consumption good and a price system is given by $q \equiv \left\{q_t^1, ..., q_t^K\right\}_{t=0}^\infty$. Agent i faces a state contingent solvency constraint, $B_{i,t}^{\xi'}(s)$, that limits security ξ' holdings at date t and $B_i \equiv \left\{B_{i,t}^1, ..., B_{i,t}^K\right\}_{t=0}^\infty$ for all i.

Given q and B_i, *agent i's problem* is

$$\max_{(c_i, a_i)} \quad E^{P_i}\left(\sum_{t=0}^\infty \rho_{i,t}\ u_i(c_{i,t})\right)$$
$$\text{s.t.} \quad \begin{cases} c_{i,t}(s) + \sum_{\xi'} q_t^{\xi'}(s)\ a_{i,t}^{\xi'}(s) = y_i(s_t) + a_{i,t-1}^{s_t}(s) & \text{for all s and t.} \\ c_{i,t}(s) \geq 0,\ a_{i,-1} = 0,\ a_{i,t}^{\xi'}(s) \geq B_{i,t}^{\xi'}(s) & \text{for all } \xi' \text{, s and t.} \end{cases}$$

Markets clear if

$$\begin{array}{lll} \sum_{i=1}^{I} c_{i,t}(s) & = y(s_t) & \text{for all s and t.} \\ \sum_{i=1}^{I} a_{i,t}^{\xi'}(s) & = 0 & \text{for all } \xi'\text{, s and t.} \end{array}$$

Definition. *A competitive equilibrium with solvency constraints (CESC) is an allocation* $\{c_i\}_{i\in\mathcal{I}}$, portfolios $\{a_i\}_{i\in\mathcal{I}}$*, a price system* q *and solvency constraints* $\{B_i\}_{i\in\mathcal{I}}$ *such that:*
(CESC 1) Given q *and* B_i, (c_i, a_i) *solves agent* i*'s problem for all* i.
(CESC 2) Markets clear.

Of course, a CESC need not be CPO (see Bloise et al [9]). In what follows, however, when we refer to CESC we always mean a CESC that is CPO. A Competitive Equilibrium (CE, hereafter) is a CESC in which the corresponding allocation is PO.

4.1 Decentralisation

Now we study the determinants of the financial wealth distribution that supports a $CESC$ allocation. First, we construct recursively the date zero-transfers needed to decentralise a CPO allocation as a time invariant function of the states (ξ, α, μ). Afterwards, we employ a properly adapted version of the Negishi's approach to pin down the CPO allocation that can be decentralised as a CESC with zero transfers.

We begin defining $A_i(\xi, \alpha, \mu)$ as the solution to the functional equation

$$A_i(\xi, \alpha, \mu) = c_i(\xi, \alpha) - y_i(\xi) + \sum_{\xi'} Q(\xi, \alpha, \mu)(\xi')\ A_i(\xi', \alpha', \mu'), \tag{11}$$

where

$$Q(\xi, \alpha, \mu)(\xi') = \max_h \left\{ \beta(\xi, \mu_h)\ \pi_{\mu_h}(\xi' \,|\xi)\ \frac{\partial u_h(c_h(\xi', \alpha'(\xi, \alpha, \mu)(\xi'))\,/\partial c_h}{\partial u_h(c_h(\xi, \alpha))\,/\partial c_h} \right\},$$

Expression (11) computes recursively the present discounted value of agent i's excess demand at the CPO allocation priced by the implicit state price $Q(\xi, \alpha, \mu)(\xi')$. Let $R^F(\xi, \alpha, \mu) = \left(\sum_{\xi'} Q(\xi, \alpha, \mu)(\xi')\right)^{-1}$ be the (implicit) risk-free interest rate

Definition. *We say that a CPO allocation generates positive risk-free interest rates if* $R^F(\xi, \alpha, \mu) > 1$ *for all* (ξ, α, μ).

Proposition 5 shows that positive risk-free interest rates guarantees that A_i is well-defined and there exist a welfare weight α_0 such that A_i is zero for every i. The allocation parameterised by α_0 is the natural candidate to be decentralised as a $CESC$.[15]

[15] In the literature studying competitive decentralisation of PO allocations in growth economies with homogeneous beliefs, the positive risk-free interest rate condition is ubiquitous to make utility levels bounded and, thus, to establish the existence of a competitive equilibrium. Since $\widehat{Q}(\xi, \mu, \alpha)(\xi') = \beta(\xi)\ \widehat{\pi}(\xi' \,|\xi) = \beta\ \pi(\xi' \,|\xi)\ g(\xi')^{1-\sigma}$ is the state price of the normalised stationary economy, the positive risk-free interest rate condition is equivalent to condition (5).

Proposition 5. *Suppose A1 holds for all agents. If the CPO allocation generates positive risk-free interest rates, there is a unique continuous function A_i solving (11). Moreover, for each (s_0, μ_0) there exists $\alpha_0 = \alpha(s_0, \mu_0) \in \mathbb{R}^I_+$ such that $A_i(s_0, \alpha_0, \mu_0) = 0$ for all i.*

We follow the Negishi's approach to decentralise the CPO allocation parameterised by α_0 as a $CESC$. For each s, t and ξ', we define recursively

$$a^{\xi'}_{i,t}(s) = A_i\left(\xi', \alpha'_{cpo}(s_t, \alpha_t(s), \mu_{s^t})(\xi'), \mu_{(s^t,\xi')}\right) \quad (12)$$

$$q^{\xi'}_t(s) = Q(s_t, \alpha_t(s), \mu_{s^t})(\xi') \quad (13)$$

$$B^{\xi'}_{i,t}(s) = A_i\left(\xi', \alpha'_{cpo}(s_t, \alpha_t(s), \mu_{s^t})(\xi'), \mu_{(s^t,\xi')}\right), \quad (14)$$

with $\mu_{s^{-1}} = \mu_0$ and α_t for $t \geq 1$ is generated by α'_{cpo} and $\alpha_0 = \alpha(s_0, \mu_0)$.

In a decentralised competitive setting with sequential trading, $A_i(s_t, \alpha_t(s), \mu_{s^t})$ can be interpreted as the financial wealth that agent i needs at date t on path s to afford the consumption bundle corresponding to the CPO allocation parameterised by $\alpha_t(s)$ given (s_t, μ_{s^t}) (see Espino and Hintermaier [17] for further discussion).[16]

Theorem 6. *Suppose that A1 holds for all agents. If the CPO allocation parameterised by $\alpha_0 = \alpha(s_0, \mu_0)$ generates positive risk-free interest rates, then it can be decentralised as a $CESC$ with portfolios $\{a_i\}_{i\in I}$, price system q and solvency constraints $\{B_i\}_{i\in I}$ defined by (12)-(14).*

4.2 The Limiting Distribution of Wealth and Consumption

Theorem 6 shows that the dynamics of the individuals' wealth and consumption in a CESC allocation is driven by the dynamic of the welfare weights. Proposition 4 shows that welfare weights have a non-degenerate limiting distribution. The following Proposition couples these two results.

Proposition 7. *Suppose $I = 2$ and A.0, A1 and A3 holds for both agents. The limiting distribution of wealth and consumption in a CESC is non-degenerate.*

An important implication of this result is that every agents' consumption is bounded away from zero regardless of whether her beliefs are correct or not (see Cao [11] for an alternative discussion). Therefore, the so-called Market Selection Hypothesis does not hold in this setting.

5 Short-Term Momentum and Long-Term Reversal

In Section 5.1 we introduce a formal definition of short-term momentum and long-term reversal in terms of the empirical autocorrelations of the equity excess returns. In Section 5.2, we argue that in any CE or CESC, the empirical autocorrelations can be approximated using the population autocorrelations. In Section 5.3 we provide a statistical characterisation of the population autocorrelations

[16] Our equilibrium concept does not rely on solvency constraints that are not too tight, see Alvarez and Jermann [3] and [4]. In our decentralisation, individual asset holdings are always at the solvency constraints by construction. However, as discussed in Alvarez and Jermann [4, pp 1131], some of these are "false corners", i.e., if the solvency constraints were marginally relaxed, the agent would not change the optimal choice of consumption and asset holdings.

in terms of the reaction of the conditional equity-premium to the realisation of the excess returns. Finally, in Section 5.4 we reinterpret the equivalent martingale measure as a market belief. We characterise the changes of the conditional equity premium to the realisation of the excess return in terms of how market pessimism changes as the market updates its belief.

5.1 Definitions

We are interested in the asset that Mehra and Prescott [27] study. Let $d_t(s)$, $p_t(s)$ and $r_t^f(s)$ be the dividend of the asset, its ex-dividend price and the (gross) risk-free interest rate, respectively, at date t on path s. For $t \geq 1$, let the one-period excess rate of return (the return hereafter) be defined as

$$r_t(s) = \frac{p_t(s) + d_t(s)}{p_{t-1}(s)} - r_{t-1}^f(s)$$

where $d_t(s) = y_t(s)$ for all t and s.

We imagine an econometrician who observes data on returns for T consecutive periods. Let

$$\overline{r}_T(s) \equiv \frac{1}{T}\sum_{t=1}^{T} r_t(s) \text{ and } \sigma_T^2 \equiv \frac{1}{T}\sum_{t=1}^{T}\left(r_t(s) - \overline{r}_T(s)\right)^2$$

be the empirical average and variance of the returns. Let

$$cov_{k,T}(s) \equiv \frac{1}{T}\sum_{t=1}^{T}\left(r_t(s) - \overline{r}_T(s)\right)\left(r_{t+k}(s) - \overline{r}_T(s)\right) \text{ and } \rho_{k,T}(s) \equiv \frac{cov_{k,T}(s)}{\sigma_T(s)\ \sigma_T(s)}$$

be the empirical autocovariance and autocorrelation coefficient of order $k \geq 1$.

Now we give a formal definition of the so-called *financial markets anomalies* that we explain.

Definition. *The asset displays short-term momentum on a path s if $\lim_{T\to\infty} \rho_{1,T}(s) > 0$. The asset displays long-term reversal on a path s if $\lim_{T\to\infty} \rho_{2,T}(s) < 0$.*

5.2 Asymptotic Approximation

The empirical autocorrelations are continuous functions of the return and (CE or $CESC$) equilibrium returns are continuous functions of a Markov process with transition function F_e on $(\Omega, \mathcal{G})$, where $e \in \{po, cpo\}$.[17] If one argues that the Markov process is ergodic with invariant distribution ψ_e, then standard arguments show that the following asymptotic approximation holds for $\tau \in \{1, 2\}$

$$\lim_{T\to\infty} cov_{\tau,T}(s) = cov^{P_e}\left(R_{1,e}, R_{\tau+1,e}\right) \qquad and \qquad \lim_{T\to\infty} \sigma_T(s) = \sigma^{P_e}\left(R_{1,e}\right), \qquad P^{\pi^*} - a.s., \tag{15}$$

where $P_e \equiv P^{F_e}(\psi_e, \cdot)$ and $R_{\tau,e} : \Omega^\infty \to \Re$ is a $\mathcal{G}_\tau-$measurable function defined by $R_{\tau,e}(\omega) \equiv R_e\left(\xi_\tau(\omega), \alpha_\tau(\omega), \mu_{\xi^\tau(\omega)}\right)\left(\xi_{\tau+1}(\omega)\right)$.

Theorem 8. *Assume A.0 holds, A.1 holds for every agent and A.2 holds for some agent. Then the asymptotic approximation (15) holds if*

(a) Allocations are PO or

(b) Allocations are CPO, $I = 2$ and both agents have dogmatic beliefs satisfying A3.

[17] When allocations are PO, with some abuse of notation, we define $\Omega \equiv S \times \Delta^1$ and $\mathcal{G}$ its $\sigma-$algebra.

Theorem 8 can be intuitively explained as follows. For the case in which allocations are PO and the dgp is iid, Beker and Espino [5] show that if A1 holds for every agent and A2 holds for some agent, then the vector of welfare weights associated with a PO allocation converges to a fixed vector almost surely. An analogous result can be proved in the case that the dgp is generated by draws from a time-homogeneous transition matrix as in this paper. This result coupled with the well-known consistency property of Bayesian learning implies the ergodicity of the Markov process with transition F_{po}. For the case in which allocations are CPO and agents have dogmatic priors satisfying $A3$, the result follows directly from Proposition 4.

Remark: The well-known result on convergence of posteriors implies that if every agent satisfies A1, there exists $\pi = (\pi_1, ..., \pi_I)$ such that μ_{i,s^t} converges weakly to μ^{π_i} for $P^{\pi^*}-$almost all $s \in S^\infty$.[18] That is, posterior beliefs converge to some dogmatic belief μ^{π_i}. Since the rest of the paper is devoted to asymptotic results, in what follows we restrict attention to the case where every agent i has a dogmatic prior μ^{π_i}. Accordingly, we omit the state variable μ^{π_i}.

5.3 Statistical Characterisation

For $\tau \geq 2$, the law of iterated expectations implies that

$$cov^{P_e}(R_{1,e}, R_{\tau,e}) = E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)\right], \tag{16}$$

where $\overline{R}_{1,e}(\omega) \equiv R_{1,e}(\omega) - E^{P_e}(R_{1,e})$ is the *abnormal return* and $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)(\omega)$ denotes the $\tau-$period ahead conditional (to the $\sigma-$algebra generated by the abnormal return) equity premium. We refer to $R_{2,e}$ and $R_{3,e}$ as the short-run and long-run returns, respectively. Likewise, $E^{P_e}\left(R_{2,e}|\,\overline{R}_{1,e}\right)(\omega)$ and $E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e}\right)(\omega)$ are the *conditional short-run equity premium and* the *conditional long-run equity premium*, respectively.

Condition (16) makes clear that the sign of the autocovariance of order τ depends on how the conditional equity premium reacts to abnormal returns. The important question is what kind of reaction of the conditional equity premium leads to short-term momentum and long-term reversal. The following definitions will be used in Proposition 9 to provide an answer to that question.

Definition. *Let $R^+ > 0$ and $R^- < 0$. For any $\tau \geq 2$,*

- $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ *trends at* (R^+, R^-) *if* $\overline{R}_{1,e}(\omega^+) = R^+$ *and* $\overline{R}_{1,e}(\omega^-) = R^-$ *implies*

$$E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)(\omega^+) > E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)(\omega^-)$$

- $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ *reverts to the mean at* (R^+, R^-) *if the reverse inequality holds.*

- $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ *trends (reverts to the mean) if,* $P_e - a.s$, $E^{P_e}\left(R_{\tau,e}|\,\overline{R}_{1,e}\right)$ *trends (reverts to the mean) at* (R^+, R^-).

[18] Where $\pi_i \in \Pi^K$ is the point in the support of agent i's prior which minimises the Kullback-Leibler divergence with respect to π^*. See the seminal work of Berk [8] for the i.i.d. case and Yamada [37] for the Markov extension.

This Proposition provides a sufficient condition for both short-term momentum and long-term reversal that follows immediately from (16) and the definition above.

Proposition 9. *If the τ-period ahead conditional equity premium trends, then the $\tau-$order autocorrelation is positive. If the τ-period ahead conditional equity premium reverts to the mean, then the $\tau-$order autocorrelation is negative. That is, (i) if the conditional short-run equity premium trends, then the asset displays short-term momentum and (ii) if the conditional long-run equity premium reverts to the mean, then the asset displays long-term reversal.*

5.4 The Economics of Predictable Returns

Note that the law of iterated expectations implies that

$$E^{P_e}\left(R_{\tau,e} \middle| \overline{R}_{1,e} \right)(\omega) = E^{P_e}\left[E^{P_e}\left(R_{\tau,e} \middle| \mathcal{G}_1 \right) \middle| \overline{R}_{1,e} \right](\omega), \text{ for all } \tau \in \{2,3\}.$$

Thus, to explain when the hypothesis of Proposition 9 are met we have to understand the behaviour of $E^{P_e}\left(R_{\tau,e} \middle| \mathcal{G}_1 \right)(\omega)$.

Definition. *Returns are unpredictable if $E^{P_e}\left(R_{\tau,e} \middle| \mathcal{G}_1 \right)(\omega)$ is $\mathcal{G}_0-$measurable for all $\tau \in \{2,3\}$.*

Returns are unpredictable if $E^{P_e}\left(R_{\tau,e} \middle| \mathcal{G}_1 \right)(\omega)$ does not change with the information released at date 1, i.e. the conditional equity premium coincides with the (unconditional) equity premium. Our next result follows immediately from (16), the definition of unpredictable return and the fact that the $\sigma-$algebra generated by the abnormal return in period 1 is contained in $\mathcal{G}_1$.

Proposition 10. *If returns are unpredictable, the asset does not display financial markets anomalies.*

The case we are interested is when returns are predictable, that is $E^{P_e}\left(R_{\tau,e} \middle| \mathcal{G}_1 \right)(\omega)$ varies with the information released at date 1. Unfortunately, this case is more complex because returns can be predictable in many different ways.

For $e \in \{po, cpo\}$, let $M_e : \mathcal{G}^\infty \to [0,1]$, be the equivalent martingale measure on $(\Omega^\infty, \mathcal{G}^\infty)$ and let $m_e : \mathcal{G} \to [0,1]$ be given by

$$m_e\left(\xi' \middle| \xi, \alpha\right) \equiv \frac{Q_e(\xi,\alpha)(\xi')}{\sum_{\tilde{\xi}\in S} Q_e(\xi,\alpha)(\tilde{\xi})} = R_e^F(\xi,\alpha)\, Q_e(\xi,\alpha)(\xi') > 0.$$

Then $M_e\left(C\left(\omega^\tau, \xi'\right) \middle| \mathcal{G}_\tau \right)(\omega) = m_e\left(\xi' \middle| \xi_\tau(\omega), \alpha_\tau(\omega)\right)$ and so m_e can be reinterpreted as the *market belief* about the states of nature next period.

The rest of this section is devoted to provide conditions on the market belief so that the conditional equity premium either trends or reverts to the mean. Note that

$$E^{P_e}\left(R_{\tau,e} \middle| \mathcal{G}_1 \right) = E^{M_e}\left(\frac{\pi_\tau^*}{m_{\tau,e}} R_{\tau,e} \middle| \mathcal{G}_1 \right) \tag{17}$$

where $m_{\tau,e}(\omega) \equiv m_e\left(\xi_\tau(\omega) \middle| \xi_1(\omega), \alpha_1(\omega)\right)$ and $\pi_\tau^*(\omega) \equiv \pi^*\left(\xi_\tau(\omega) \middle| \xi_1(\omega)\right)$. When the ratio $\frac{\pi_\tau^*}{m_{\tau,e}}$ is identically equal to one, then the $\tau-$period ahead conditional equity premium is equal to zero. When the ratio $\frac{\pi_\tau^*}{m_{\tau,e}}$ is different from one with positive probability, typically, the $\tau-$period ahead conditional equity premium is different from zero. For example, if the ratio $\frac{\pi^*_\tau}{m_{\tau,e}}$ is greater than one when the return is positive and smaller than one otherwise, a situation where we say the market belief is pessimistic about the return, then the $\tau-$period ahead conditional equity premium is positive.

In the PO and CPO allocations we consider later, the sign of the return at any given period depends only on the state of nature that period. The following definition gives a useful taxonomy of states when that property holds.

Definition. *A state of nature ξ' is good news if, $\psi_e - a.s$, $R_e(\xi,\alpha)(\xi') > 0$. A state of nature ξ' is bad news if, $\psi_e - a.s$, $R_e(\xi,\alpha)(\xi') < 0$.*

Condition (17) makes clear that the sign of $E^{P_e}(R_{\tau,e}|\mathcal{G}_1)$ depends on how $\frac{\pi^*_\tau}{m_{\tau,e}}$ depends on the state of nature at τ. However, to understand trending and mean reversion what we actually need to understand is how $E^{P_e}(R_{\tau,e}|\mathcal{G}_1)$ changes as the conditioning information changes. With that purpose in mind, we introduce the following definition.

Definition. *The market is more pessimistic at state (ξ^+,α^+) than at state (ξ^-,α^-) if for every bad news ξ',*

$$\frac{\pi^*(\xi'|\xi^+)}{m_e(\xi'|\xi^+,\alpha^+)} < \frac{\pi^*(\xi'|\xi^-)}{m_e(\xi'|\xi^-,\alpha^-)}$$

where the ratio $\frac{\pi^*(\xi'|\xi)}{m_e(\xi'|\xi,\alpha)}$ denotes the market pessimism at (ξ,α).

In the i.i.d. case, i.e. $\pi^*(\xi'|\xi)$ is independent of ξ, the condition for the market belief to be more pessimistic at (ξ^+,α^+) than at (ξ^-,α^-) reduces to $m_e(\xi'|\xi^+,\alpha^+) > m_e(\xi'|\xi^-,\alpha^-)$.

The following definition let us move from the returns space to the state space. This is necessary because Proposition 9 applies to random variables defined on the former while $E^{P_e}(R_{\tau,e}|\mathcal{G}_1)(\omega)$ and the market belief are defined on the latter.

Definition. *State $(\tilde{\xi},\tilde{\alpha})$ is consistent with realisation R of $\overline{R}_{1,e}$ if there exists a state (ξ,α) such that $\overline{R}_e(\xi,\alpha)(\tilde{\xi}) = R$ and $\alpha'_e(\xi,\alpha)(\tilde{\xi}) = \tilde{\alpha}$.*

To grasp the difficulty in finding a sufficient condition for trending, consider the case in which, given (ξ,α), the range of $R_e(\xi,\alpha)(\cdot)$ has only two elements, say $R_e(\xi,\alpha)(L)$ and $R_e(\xi,\alpha)(H)$ and H is good news. $E^{P_e}\left(R_{2,e}|\overline{R}_{1,e}\right)(\omega)$ trends at (R^+,R^-) if

$$\frac{\pi^*(L|H)R_e(H,\alpha^+)(L) + \pi^*(H|H)R_e(H,\alpha^+)(H)}{\pi^*(L|L)R_e(L,\alpha^-)(L) + \pi^*(H|L)R_e(L,\alpha^-)(H)} > 1$$

for states (H,α^+) and (L,α^-) consistent with realisations R^+ and R^-, respectively, of $\overline{R}_{1,e}$. Note that even if $R_e(H,\alpha^+)(\xi') > R_e(L,\alpha^-)(\xi')$ for all $\xi' \in \{L,H\}$, mean reversion might arise if $\pi^*(L|H)$ is sufficiently larger than $\pi^*(L|L)$. So, we need to find an expression that relates trending with the changes in $\pi^*(L|\xi)$ and $R_e(\xi,\alpha)(L)$ induced by changes in the state (ξ,α).

Since $E^{M_e}(R_{\tau+1,e}|\mathcal{G}_\tau)(\omega) = 0$, then we can write

$$R_e(\xi_\tau,\alpha_\tau)(L) = -\frac{1-m_e(L|\xi_\tau,\alpha_\tau)}{m_e(L|\xi_\tau,\alpha_\tau)}R_e(\xi_\tau,\alpha_\tau)(H) \tag{18}$$

Therefore, using (18)

$$E^{P_e}\left(R_{\tau+1,e}|\,\mathcal{G}_\tau\right)=\left(1-\frac{\pi^*\left(L|\,\xi_\tau\right)}{m_e\left(L\,|\xi_\tau,\alpha_\tau\right)}\right)R_e\left(\xi_\tau,\alpha_\tau\right)(H)$$

Thus, $E^{P_e}\left(R_{2,e}|\,\overline{R}_{1,e}\right)$ trends at (R^+,R^-) if and only if

$$\left(1-\tfrac{\pi^*(L|H)}{m_e(L|H,\alpha^+)}\right)R_e\left(H,\alpha^+\right)(H)>\left(1-\tfrac{\pi^*(L|L)}{m_e(L|L,\alpha^-)}\right)R_e\left(L,\alpha^-\right)(H) \tag{19}$$

for all states (H,α^+) and (L,α^-) consistent with R^+ and R^-, respectively.

Definition. *The recession bias effect at* (α^+,α^-) *is*

$$\Upsilon_e\left(\alpha^+,\alpha^-\right)\equiv\frac{\pi^*\left(L|\,L\right)}{m_e\left(L\,|L,\alpha^-\right)}-\frac{\pi^*\left(L|\,H\right)}{m_e\left(L\,|H,\alpha^+\right)}.$$

The recession bias effect will be useful to characterise the reaction of the conditional equity-premium in allocations where the market belief overestimates the likelihood of a recession at every state (ξ,α). In those allocations, the recession bias effect is positive if and only if the market is more pessimistic at the expansion state than at the recession state and it is negative otherwise.

Definition. *The return inertia effect at* (α^+,α^-) *is*

$$I_e\left(\alpha^+,\alpha^-\right)\equiv R_e\left(H,\alpha^+\right)(H)-R_e\left(L,\alpha^-\right)(H).$$

The return inertia effect is positive if and only if the highest return in the following period is larger after expansions than after recessions. It follows from (19) that the conditions below are sufficient for $E^{P_e}\left(R_{2,e}|\,\overline{R}_{1,e}\right)$ to trend at (R^+,R^-):

$$\text{(i) }\Upsilon_e\left(\alpha^+,\alpha^-\right)>0\qquad\text{(ii) }I_e\left(\alpha^+,\alpha^-\right)>0, \tag{20}$$

where (H,α^+) and (L,α^-) are consistent with realisations R^+ and R^-, respectively, of $\overline{R}_{1,e}$. We refer to conditions (i) and (ii) as *positive recession bias* and *positive return inertia* effects, respectively.

6 Financial Market Anomalies?

In this section we evaluate qualitatively and quantitatively the ability of CE and CESC allocations to generate short-term momentum and long-term reversal.

Property A captures and intuitive relationship between short-term momentum, long-term reversal and the persistency of expansions.

PROPERTY A: If the asset displays short- term momentum, then it also displays long-term reversal if and only if expansions are not strongly persistent.

The following Theorem, see Beker and Espino [7], gives conditions for Property A to hold.[19]

Theorem 11. *Suppose A0 holds and either A1.a or A1.b holds for every agent. Property A holds in*
(a) Any CE of a baseline growth economy where A2.a or A2.b holds for some agent.
(b) Any CESC of a baseline growth economy where every agent has homogeneous dogmatic beliefs satisfying A2 and $\underline{\alpha}_1(1)=\underline{\alpha}_1(2)$.

[19] For the sake of completeness we provide a proof in the supplementary material.

6.1 Asset returns with Aggregate Growth

To facilitate quantitative analysis, in this section we define asset returns for a growth economy with dogmatic beliefs. The particular form of the growth process we assume in (4) makes the state prices in the growth economy, Q_e, independent of current output. Indeed,[20]

$$Q_e(\xi,\alpha)(\xi') = \beta \ \max_h \left\{ \pi_h(\xi' \,|\xi\,) \ \frac{(\widehat{c}_h(\xi',\alpha_e'(\xi,\alpha)(\xi'))^{-\sigma}}{(\widehat{c}_h(\xi,\alpha))^{-\sigma}} \right\} \ g(\xi')^{-1} = \widehat{Q}_e(\xi,\alpha)(\xi') \ g(\xi')^{-1}$$

The price-dividend ratio in the growth economy, P_e^D, is the solution to

$$\begin{aligned} P_e^D(\xi,\alpha) &= \sum_{\xi'} Q_e(\xi,\alpha)(\xi') g(\xi') \ \left(1 + P_e^D(\xi',\alpha_e'(\xi,\alpha)(\xi'))\right) \\ &= \sum_{\xi'} \widehat{Q}_e(\xi,\alpha)(\xi') \ \left(1 + P_e^D(\xi',\alpha_e'(\xi,\alpha)(\xi'))\right) \end{aligned}$$

for all (ξ,α). Therefore, the asset return can be written as

$$R_e^E(\xi,\alpha)(\xi') = \frac{1 + P_e^D(\xi',\alpha_e'(\xi,\alpha)(\xi'))}{P_e^D(\xi,\alpha)} g(\xi').$$

6.2 Calibration

We set $S = 4$ to allow for both aggregate and idiosyncratic risk while respecting symmetry across agents. We specify the endowment process with four values for the income of each agent and two values for the growth rate. Even and odd states correspond to high and low, respectively, growth rates. Agent 1's income share is high in state 1 and 2 and low otherwise. Because of symmetry, there are 10 parameters to be selected: six for π^*, two for $y_1(\cdot)$ and two for $g(\cdot)$. We calibrate these 10 free parameters using the same 10 moments describing the US aggregate and household income data that Alvarez and Jermann use (see Appendix C for the calibrated parameters.)

In Table 1 we report our computations of the annual averages of the risk-free interest rate and equity-premium and also the 1^{st} and 2^{nd} order empirical annual autocorrelations for the US stock market using Mehra and Prescott [27] dataset .

Table 1

Period	*Risk-Free Rate*	*Equity-Premium*	$1^{st}-order$ *autocorrelation*	$2^{nd}-order$ *autocorrelation*
1900-2000	*0.8%*	*6.18%*	*13.94%*	*-15.28%*

If a model generates a time series of returns that displays both short-term momentum and long-term reversal for some values of β and σ, we say that it's predictions are *qualitatively* accurate. Afterwards, we set β and σ to match the average annual risk-free rate of 0.8% and equity premium of 6.18%. If for those parameters the model can generate autocorrelations that are both of the same sign and order of magnitude as those in Table 1, we say that it's predictions are *quantitatively* accurate.

[20]Note that (5) reduces to $\max_{\xi'} \left\{ \beta \sum_{\xi'} \pi_i(\xi' \,|\xi) g(\xi')^{1-\sigma} \right\} < 1$.

6.3 Standard Model I: CE Allocations

In this section we present our numerical solutions for CE. Figure 1 plots the 1^{st} and 2^{nd}-order autocorrelations of returns for $\sigma \in [0, 18]$.

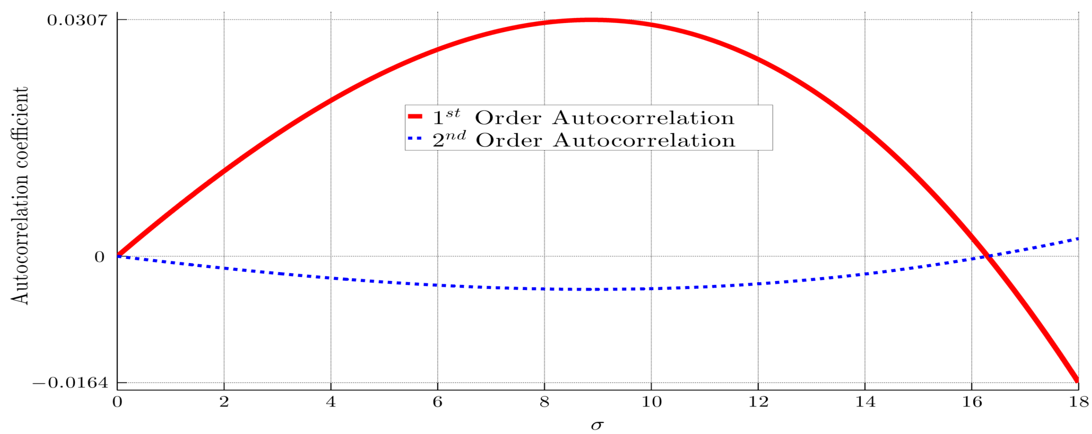

Figure 1: Autocorrelation coefficients in CE

The salient feature of Figure 1 is that, contrary to the conventional wisdom, the asset can actually display short-term momentum and long-term reversal in CE. Indeed, the signs of the autocorrelations are as in Table 1 if σ is below a threshold, which is close to 16 in our example.

To explain the sign of these correlations, we first recall that Theorem 11(a) shows that short-term momentum implies long-term reversal. Thus, we need to explain the sign of the 1^{st} order autocorrelations, i.e. why short-term momentum occurs, and, by Proposition 9, it suffices to argue that trending and mean reversion occurs for low and high values, respectively, of σ. Since returns in CE depend only on the current state of nature and the growth rate, its range takes only two values given the current state of nature. Therefore, we can use the decomposition we introduce in (20) to explain why the conditional short-run equity premium trends.

We refer to events $\{1, 3\}$ or $\{2, 4\}$ as a recession (L) or an expansion (H), respectively. Note that, under our assumptions, any expansion state is good news while any recession one is bad news.

We abuse notation and denote the risk-free rate by $R^F_{po}(\tilde{\xi}, \alpha_\infty)$ for $\tilde{\xi} \in \{L, H\}$. Then,

$$R^F_{po}(\tilde{\xi}, \alpha_\infty) = \left[\beta\left(\pi^*(L|\,\tilde{\xi})g\,(L)^{-\sigma} + \pi^*(H|\,\tilde{\xi})g\,(H)^{-\sigma}\right)\right]^{-1}$$

where $\pi^*\left(L|\,L\right) \equiv \pi^*\left(1|\,1\right) + \pi^*\left(3|\,1\right)$ and $\pi^*\left(L|\,H\right) \equiv \pi^*\left(1|\,2\right) + \pi^*\left(3|\,2\right)$. Since $\pi^*\left(L|\,H\right) > \pi^*\left(L|\,L\right)$, then the risk free rate is larger in recessions than in expansions for all $\sigma > 0$. That is

$$R^F_{po}(L, \alpha_\infty) = R^F_{po}(H, \alpha_\infty) \text{ if } \sigma = 0 \text{ and } R^F_{po}(L, \alpha_\infty) > R^F_{po}(H, \alpha_\infty) \text{ if } \sigma > 0$$

The market belief about a recession at $(\tilde{\xi}, \alpha_\infty)$ is

$$m_{po}(L|\tilde{\xi}, \alpha_\infty) = R^F_{po}(\tilde{\xi}, \alpha_\infty)\ \beta\ \pi^*(L|\tilde{\xi})\ g\,(L)^{-\sigma}$$

and, therefore, market pessimism is inversely proportional to the risk-free rate.

In Figure 2 we plot the recession bias (on the left-hand side) and the return inertia effects (on the right-hand side) for different levels of risk aversion.

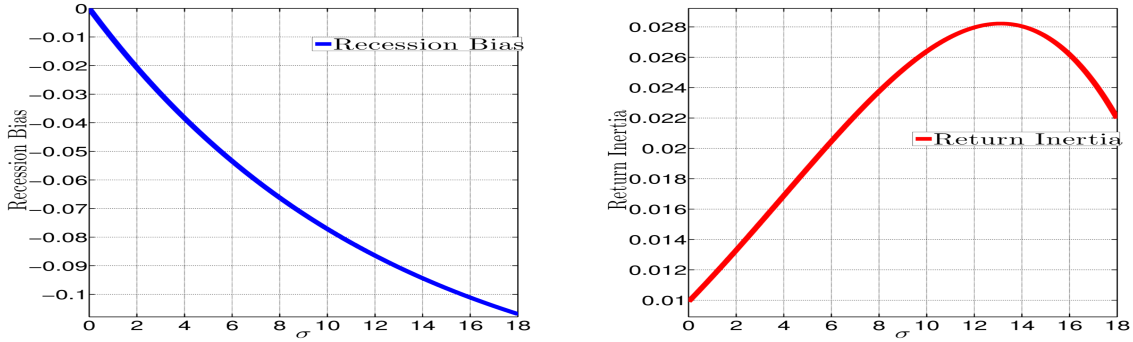

Figure 2: Recession Bias and Return Inertia Effects in CE

Figure 2 shows the recession bias effect is negative and the return inertia effect is positive for all σ. For low levels of σ, there is trending because the recession bias effect vanishes and the return inertia effect is small but positive. For high levels of σ, there is mean reversion because the recession bias effect becomes strongly negative and outweighs the positive return inertia effect.

First, we explain the recession bias effect. For $\sigma = 0$, the recession bias effect is zero because interest rates are state independent and so is the market pessimism. For $\sigma > 0$, the recession bias effect becomes negative because market pessimism is a decreasing function of the risk-free rate which implies that

$$\frac{\pi^*(L|L)}{m_{po}(L|L,\alpha_\infty)} < \frac{\pi^*(L|H)}{m_{po}(L|H,\alpha_\infty)} \text{ for any } \sigma > 0.$$

To explain the return inertia effect, we plot it together with the equity return and the risk-free rate in Figure 3.

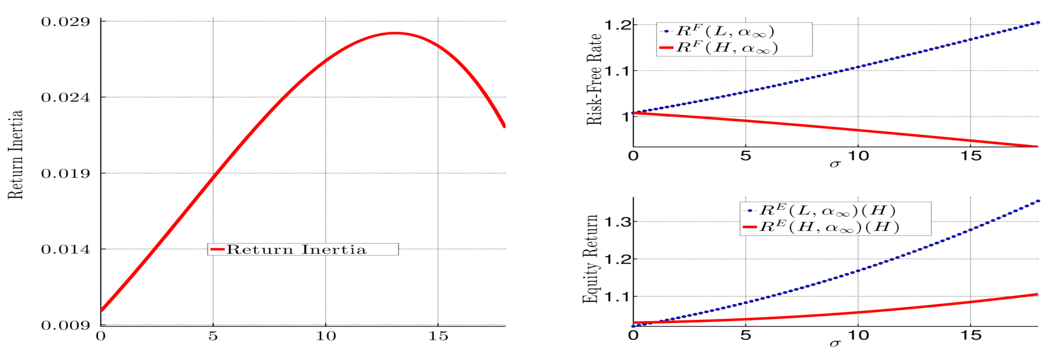

Figure 3: Determinants of the Return Inertia Effect in CE

Note that when $\sigma = 0$, $R^E_{po}(\cdot,\alpha_\infty)(H)$ is larger after an expansion than after a recession while the risk free rate is state independent. Therefore, $R^E_{po}(\cdot,\alpha_\infty)(H)$ is larger after an expansion than after a recession for $\sigma = 0$ and, by continuity, for low levels of σ as well. As σ increases, the situation does not change in spite of $R^E_{po}(H,\alpha_\infty)(H) - R^E_{po}(L,\alpha_\infty)(H)$ being negative because $R^F_{po}(L,\alpha_\infty) - R^F_{po}(H,\alpha_\infty)$ is positive and larger in absolute value.

The left and right hand sides of Figure 4 illustrates the equity premium and the values of β required to fit the risk-free rate of 0.8%, respectively, as a function of σ (see Appendix C.1).

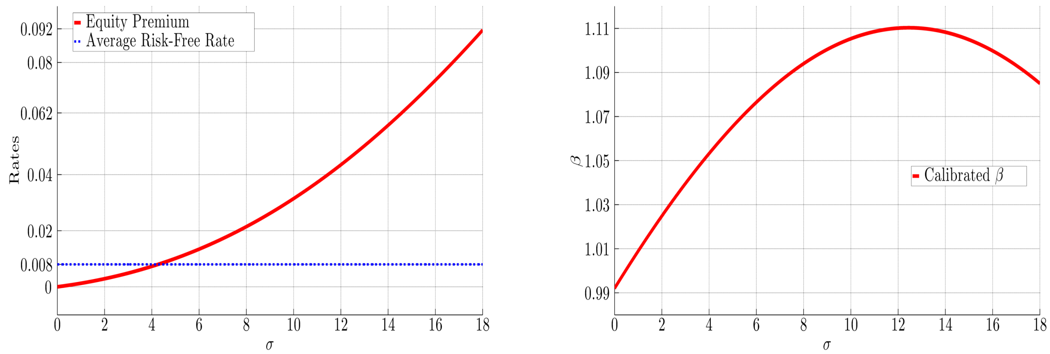

Figure 4: Equity Premium and Calibrated β in CE

The left-hand side of Figure 4 shows that a high σ, close to 14.5 in our calibration, is necessary to obtain an equity-premium of 6.18% as Mehra and Prescott [27] pointed out. The right-hand side of Figure 4 makes it clear that β has to be larger than 1, as put forward by Weil [36].

Figure 1 forcefully show that in spite of the conventional wisdom, CE allocations are able to generate qualitatively accurate predictions. In particular, short-term momentum is due to positive return inertia effect. However, for the values of β and σ for which the model matches the historical averages of the risk free rate and equity premium, the autocorrelations have the same sign as in Table 1 but they are negligible. Indeed, in Figure 1 we can see that for σ close to 14.5 both magnitudes are lower than 1% in absolute value. We conclude that the failure of CE to explain short-term momentum and long-term reversal is only of a quantitative nature.

6.4 Standard Model II: CESC Allocations-Homogeneous Beliefs

In this section we report the results of our numerical simulations of CESC when both agents have correct beliefs. Figure 5 shows the autocorrelations of asset returns in CESC for values of $\sigma \in [2, 5]$ when β is chosen to fit the average risk-free rate of 0.8%. In addition, it compares them with the corresponding autocorrelations in the CE of the same economy.

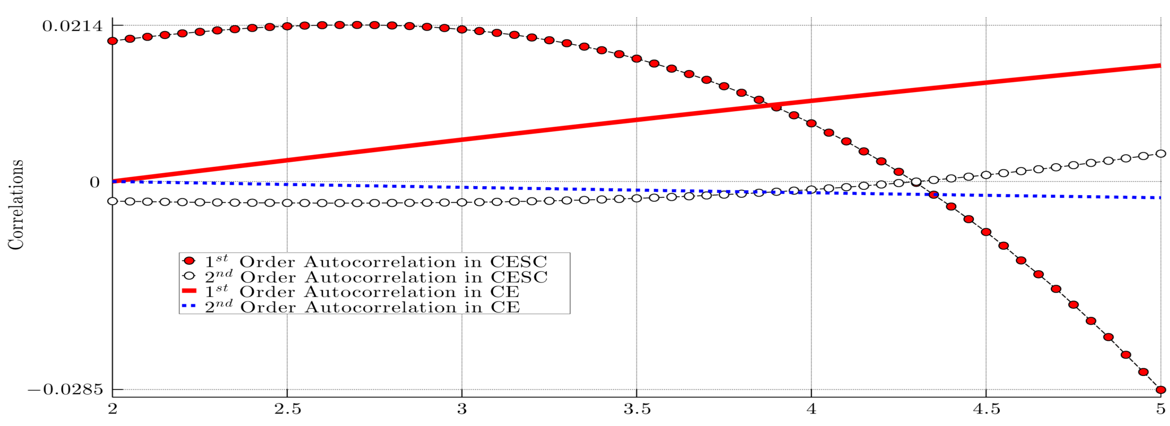

Figure 5: Autocorrelation Coefficients in CESC with correct beliefs.

There are two important lessons to draw from Figure 5. First, the asset displays short-term momentum (and long-term reversal) only for moderate levels of σ. Indeed, the 1^{st}-order autocorrelation is positive only for $\sigma < 4.3$. Second, CESC allocations generate larger (in absolute value) autocorrelations, both in the short-run as well as in the long-run, than CE allocations for moderate levels of σ while the opposite occurs for large levels of σ. Indeed, the absolute value of the autocorrelations are larger in a CESC than in a CE only for $\sigma < 3.4$.

To explain the sign of these autocorrelations, we consider the CESC of a proxy economy that differs from the original economy in that its minimum enforceable weights in states 2 and 4 are set equal to those of the original economy in states 1 and 3. We use the symbol ˜ above a variable when the latter belongs to the proxy economy. That is,

$$
\begin{aligned}
\tilde{\underline{\alpha}}_{i,cpo}(\xi) &= \underline{\alpha}_{i,cpo}(\xi) \qquad \text{if } \xi \in \{1,3\}, \\
\tilde{\underline{\alpha}}_{1,cpo}(2) &= \underline{\alpha}_{1,cpo}(1), \\
\tilde{\underline{\alpha}}_{2,cpo}(4) &= \underline{\alpha}_{2,cpo}(3).
\end{aligned}
$$

The numerical solutions for both economies are similar because the invariant distribution (see Appendix C.2) places very little mass on $\left[\underline{\alpha}_{1,cpo}(2), \underline{\alpha}_{1,cpo}(1)\right)$ and $\left(\underline{\alpha}_{2,cpo}(4), \underline{\alpha}_{2,cpo}(3)\right]$. Since the minimum enforceable weights of the proxy economy are measurable with respect to the growth rate, both the market belief as well as the returns inherit that property. Furthermore, any expansion state is good news and any recession state is bad news. Since the growth rate takes only two values, so does the return given the current state, and so we are able to use the decomposition introduced in (20).

Note that the support of the invariant distribution of the (proxy economy) welfare weights consists of $(\tilde{\underline{\alpha}}_{1,cpo}(1), 1-\tilde{\underline{\alpha}}_{1,cpo}(1))$ and $(1-\tilde{\underline{\alpha}}_{2,cpo}(2), \tilde{\underline{\alpha}}_{2,cpo}(2))$. Therefore, without loss in generality, returns can be written as a function only of the state of nature. With some abuse of notation we denote the the risk-free rate of the proxy economy in state of nature $\tilde{\xi} \in \{L, H\}$ as

$$\tilde{R}^F_{cpo}(\tilde{\xi}) \equiv \beta^{-1}\left(\tilde{\pi}(L|\tilde{\xi})g(L)^{-\sigma} + \tilde{\pi}(H|\tilde{\xi})g(H)^{-\sigma}\right)^{-1}$$

where $\tilde{\pi}(L|L) \equiv \pi^*(1|1) + \pi^*(3|1)\theta_{AJ}$, $\tilde{\pi}(L|H) \equiv \pi^*(1|2) + \pi^*(3|2)\theta_{AJ}$ and the term $\theta_{AJ} > 1$, defined in Appendix C.2, reflects the increase in the marginal valuation of consumption when, as in Alvarez and Jermann [3], the enforceability constraint binds. Since the marginal valuation of consumption is highest in a recession and the latter is more likely after an expansion has occurred, i.e. $\tilde{\pi}(L|H) > \tilde{\pi}(L|L)$, then the risk-free rate of the proxy economy satisfies:

$$\tilde{R}^F_{cpo}(L) > \tilde{R}^F_{cpo}(H)$$

The market belief of the proxy economy adjusts upwards the probability of those states where the enforceability constraint binds and it is always the agent who is rich in that state who is constrained. Since the enforceability constraints that bind are those that correspond to states next period where the income shares change with respect to the current one, the market belief of the proxy economy is

$$\tilde{m}_{cpo}(\xi'|\xi) = \begin{cases} \beta\ \tilde{R}^F_{cpo}(\xi)\ \pi^*(\xi'|\xi)\ g(\xi')^{-\sigma} & \text{if } \epsilon_1(\xi') = \epsilon_1(\xi') \\ \beta\ \tilde{R}^F_{cpo}(\xi)\ \pi^*(\xi'|\xi)\ g(\xi')^{-\sigma}\ \theta_{AJ} & \text{if } \epsilon_1(\xi') \neq \epsilon_1(\xi') \end{cases}$$

Theorem 11(b) implies that it suffices to explain when short-term momentum occurs in the proxy economy. By Proposition 9, it suffices to argue that trending and mean reversion occurs for low and high values, respectively, of σ.

Note that market pessimism in state of nature $\tilde{\xi} \in \{L, H\}$ is

$$\frac{\pi^*(L|\tilde{\xi})}{\tilde{m}_{cpo}(L|\tilde{\xi})} = \frac{1}{\beta(\sigma)\tilde{R}^F_{cpo}(\tilde{\xi})} \frac{\pi^*(L|\tilde{\xi})}{\tilde{\pi}(L|\tilde{\xi})}\, g\,(L)^{\sigma}$$

where $\tilde{m}_{cpo}\,(L\,|L\,) \equiv \tilde{m}_{cpo}\,(1\,|1\,) + \tilde{m}_{cpo}\,(3\,|1\,)$ and $\tilde{m}_{cpo}\,(L\,|L\,) \equiv \tilde{m}_{cpo}\,(2\,|2\,) + \tilde{m}_{cpo}\,(4\,|2\,)$.

Figure 6 shows that the recession bias effect is negative and decreasing in σ but the return inertia effect is always positive.

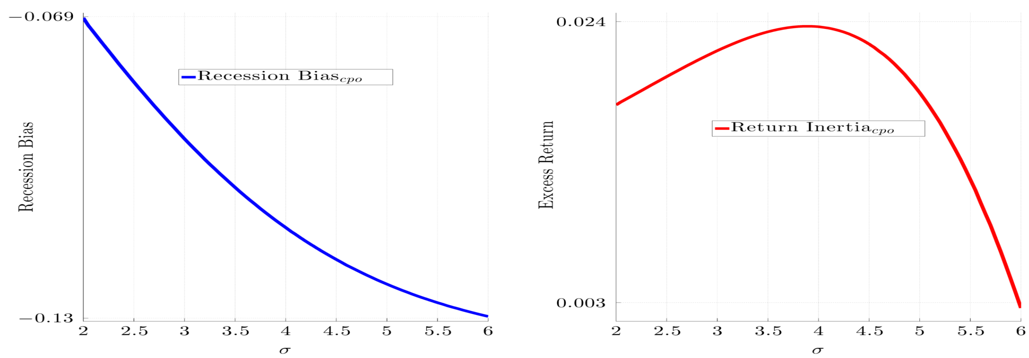

Figure 6: Recession Bias and Return Inertia Effects in CESC with correct beliefs

Consequently, for low σ there is trending because the return inertia effect dominates. For high σ, there is mean reversion because the return inertia effect vanishes.

To understand why the recession bias effect is negative, note that since $\pi^*\,(3\,|1\,)$ and $\pi^*\,(4\,|2\,)$ are small, then $\frac{\pi^*(L|\tilde{\xi})}{\tilde{\pi}(L|\tilde{\xi})}$ is close to one and so market pessimism is driven by the risk-free rate, that is

$$\frac{\pi^*\,(L\,|L\,)}{\tilde{m}_{cpo}\,(L\,|L\,)} < \frac{\pi^*\,(L\,|H\,)}{\tilde{m}_{cpo}\,(L\,|H\,)}$$

To understand why the return inertia effect is positive, we plot the return inertia effect together with the equity return and the risk free rate in Figure 7.

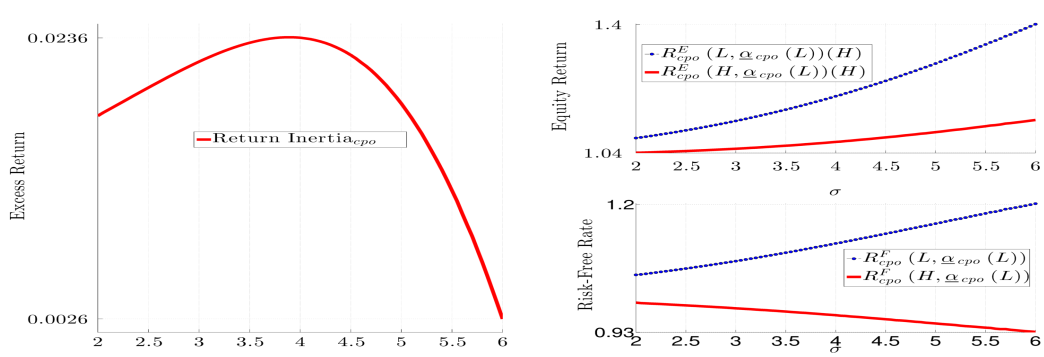

Figure 7: Return Inertia Effect in CESC with correct beliefs.

Note that even though $R^E_{cpo}(\cdot,\cdot)(H)$ is larger after a recession than after an expansion, the differential of the risk-free rate between a recession and an expansion is much larger and dominates.

The right-hand side of Figure 8 shows the values of β required to fit the risk-free rate of 0.8% for different values of σ. The left-hand side of Figure 8 shows how the equity premium changes with σ.

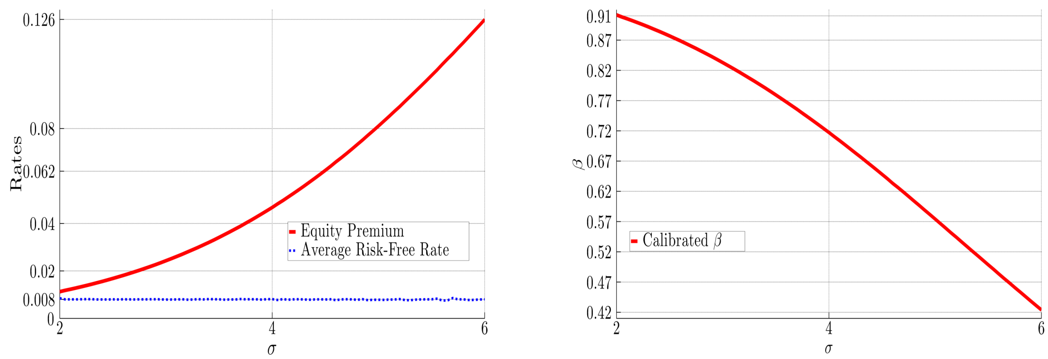

Figure 8: Equity Premium and Calibrated β in CESC with correct beliefs

The right-hand size of Figure 8 shows that $\beta < 1$. The left-hand side of Figure 8 shows that only values of σ close to 4.5 generate an equity-premium close to 6.18%.

Figure 5 makes clear that CESC allocations are able to generate predictions that are qualitatively accurate. For those parameterizations, short-term momentum is due to a positive return inertia effect. Figure 5 also shows that for the values of β and σ for which the model matches the historical averages of the risk free rate and equity premium, the model fails to generate short-term momentum and long-term several. Consequently, the failure of CESC allocations is only of a quantitative nature.

6.5 CESC Allocations: Heterogeneous Beliefs

In this section we report the results of our numerical simulations for CESC when agents have heterogeneous beliefs. We first report the autocorrelations of returns for the calibrated economy and then we explain the role played by belief heterogeneity.

We assume agent 1 has correct beliefs and agent 2 believes the transition function belongs to the following family of transition matrices parameterised by $\varepsilon \in (-\pi^*(1|2), \pi^*(2|2))$:

$$\pi^* + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \varepsilon & -\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -\varepsilon \end{bmatrix}$$

In this parameterization, agent 2 has (possibly) incorrect beliefs regarding the persistency of an expansion, i.e. $\pi_2(2|2) = \pi_2(4|4) = \pi^*(2|2) - \varepsilon = 0.682 - \varepsilon$, and correct beliefs otherwise. In particular, he has correct beliefs regarding the idiosyncratic state. Clearly, this parameterization satisfies assumption A3 for every $\epsilon > 0$.

Figure 9 plots the 1^{st} and 2^{nd}-order autocorrelations as a function of agent 2's beliefs. For each value of $\pi_2\left(2|\,2\right)$, β and σ are calibrated to match both the risk-free rate of 0.8% and the equity premia of 6.18%. Pessimism generates autocorrelations of the same order of magnitude than those in Table 1. For example, 1^{st} and 2^{nd}-order autocorrelations are 0.14 and -0.083, respectively, when $\pi_2\left(2|\,2\right) = 0.222$, $\sigma = 2$ and $\beta = 0.767$.

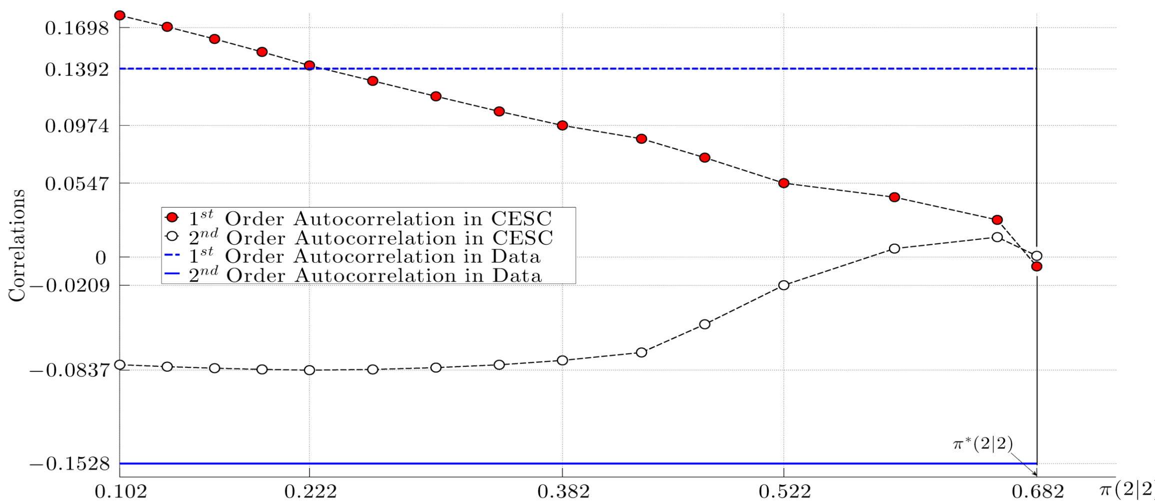

Figure 9: Autocorrelation Coefficients in the fully calibrated CESC - Heterogeneous beliefs.

One of the difficulties to explain the effect of belief heterogeneity on the autocorrelations of the calibrated economy in Figure 9 is that as agent 2's beliefs change, both σ and β change. In order to isolate the effect of belief heterogeneity from that induced by the changes in σ and β, Figure 10 plots the autocorrelations for fixed values of σ and β. In particular, we consider the high-β parameterization of Alvarez and Jermann [4], i.e., $\sigma = 3.5$ and $\beta = 0.7794$. In that setting, Figure 10 underscores that as agent 2 becomes more pessimistic, the asset displays autocorrelations of an order of magnitude larger than those predicted by the previous models.

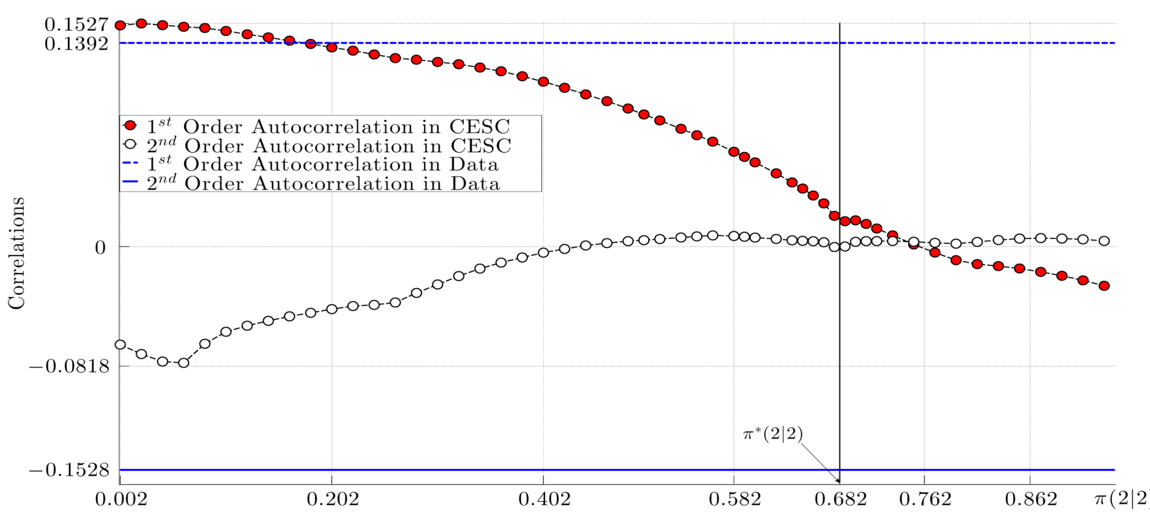

Figure 10: Autocorrelations - AJ High-β parameterization

To understand the effect of belief heterogeneity on the market belief, we first consider the role it plays on the law of motion of the welfare weights. Figure 11 compares the law of motion of the welfare weights of the heterogeneous beliefs economy (on the left, where $\pi_2(2|2) = 0.382$) to that of the homogeneous beliefs economy (on the right, where $\pi_2(2|2) = \pi_2^*(2|2) = 0.682$).

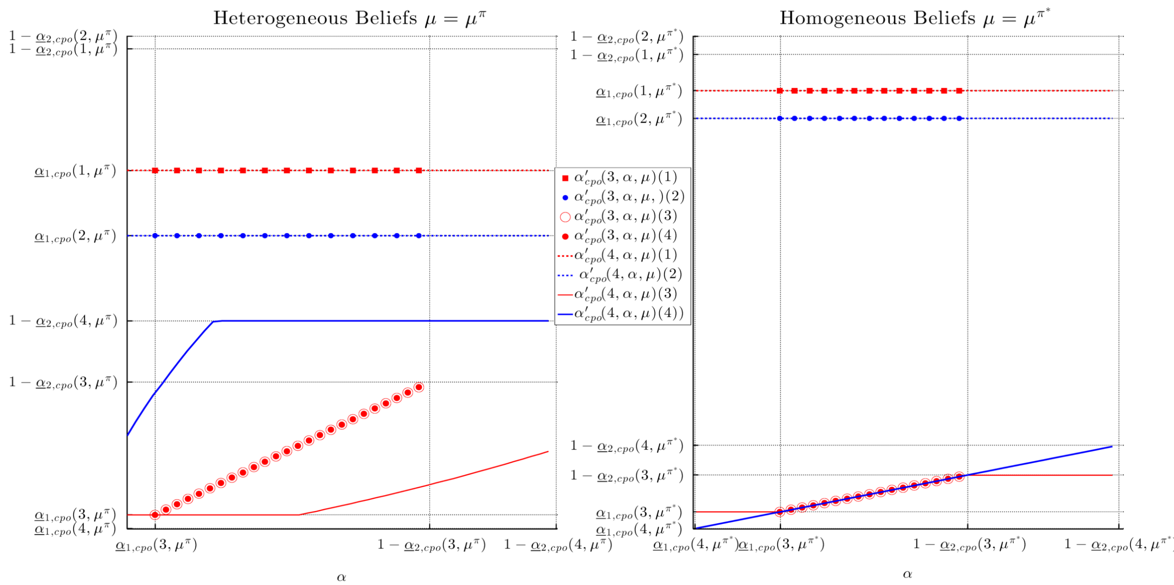

Figure 11: Welfare Weights Law of Motion

For simplicity, we only consider the law of motion evaluated on welfare weights of states 3 and 4. There are two similarities with the case of homogeneous beliefs. First, if one transits from an state where agent i has low income share to an state where agent i's has high income share, i's enforceability constraints bind at those states and so it is always agent $-i$ who is unconstrained.[21] Second, if one transits from an state where agents have homogeneous beliefs (states 1 or 3) to an state with the same income share as today's, then the welfare weight does not change and no enforceability constraint binds.[22] The difference with the case of homogeneous beliefs arises when one transits from an state where there is belief heterogeneity (states 2 or 4) to a state with the same income shares (states 1 or 3, respectively).[23] This is because belief heterogeneity makes the welfare weights of the agent who puts lower probability on tomorrow's state to decrease and so it might make his enforceability constraint to bind. Most importantly, it is always the agent who puts higher probability on tomorrow's state who is unconstrained. In particular, when the transition is from an expansion to a recession (with no reversal of income shares) and $\epsilon > 0$, it is always agent 2 who is unconstrained.

Figure 12 plots the conditional invariant distribution of the welfare weights for different values of

[21] That is, the enforceability constraints of states 3 and 4 bind if the current state is 1 or 2 and that of states 1 and 2 bind if the current state is 3 or 4.

[22] For our parameterization of agent 2's pessimism, the support of the invariant distribution does not include the intervals $(\underline{\alpha}_1(4), \underline{\alpha}_1(3))$ and $(\underline{\alpha}_1(2), \underline{\alpha}_1(1))$ since they are not contained in the range of the law of motion. Therefore, our discussion of the law of motion refers always to welfare weights outside those sets.

[23] That is, from states 2 or 4 to states in $\{1, 2\}$ or in $\{3, 4\}$, respectively.

agent 2's pessimism. The existence of an invariant distribution follows by Proposition 4.

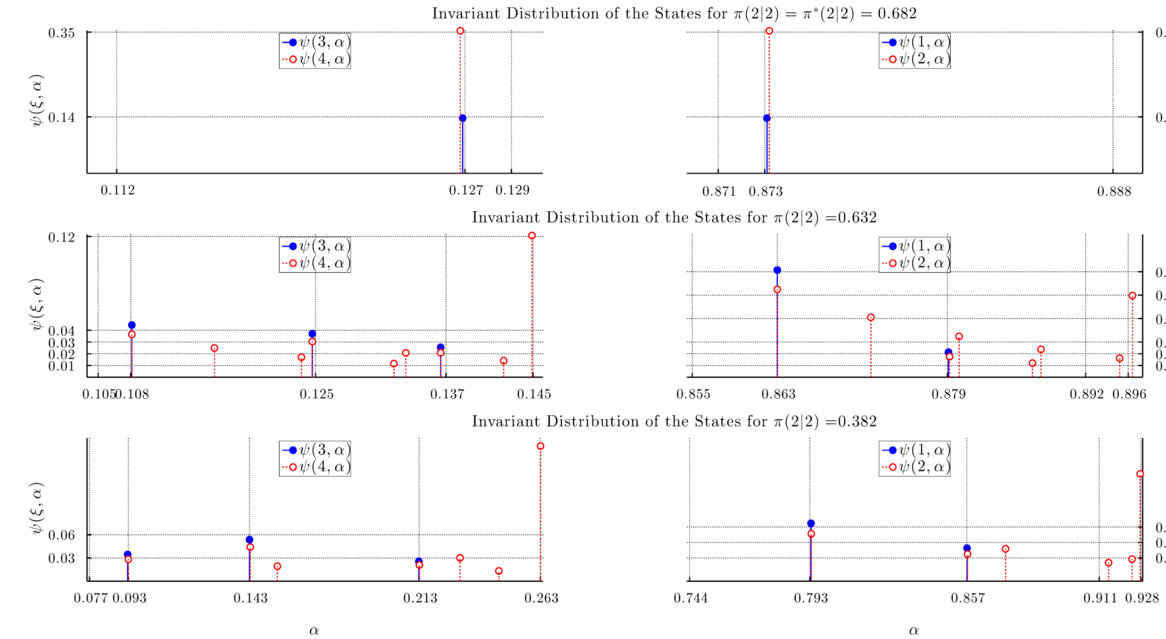

Figure 12: Invariant Distribution of CPO Welfare Weights

As agent 2 becomes more pessimistic, the conditional invariant distribution not only becomes non-degenerate but it also places more mass on larger values of agent 1's welfare weights.

Figure 13 plots the distribution of the conditional equity premium.

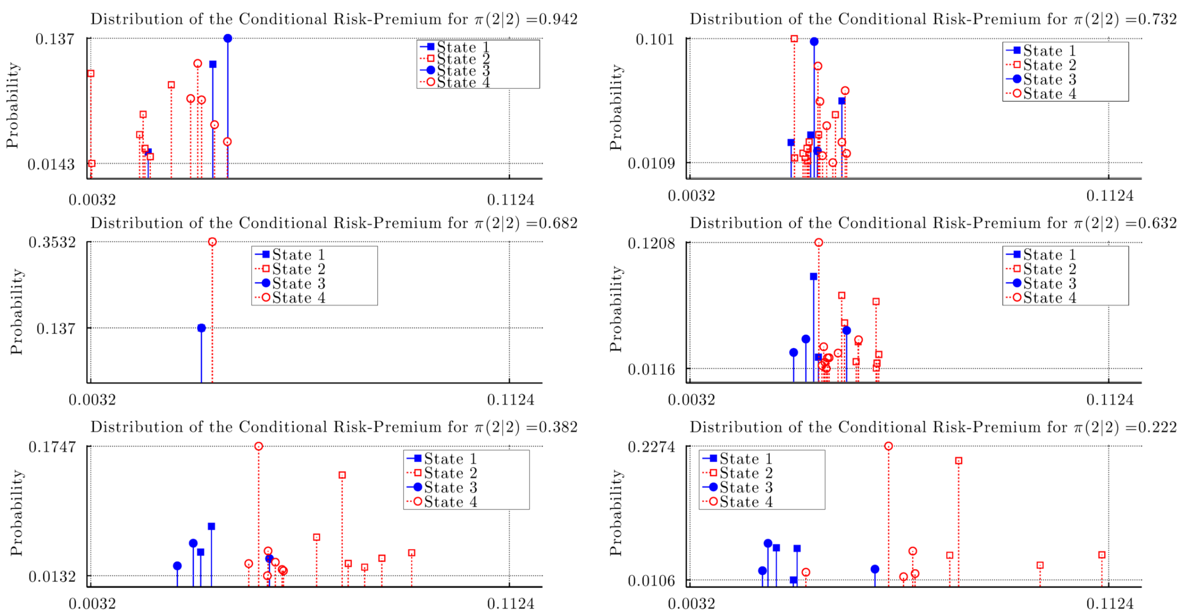

Figure 13: Distribution of the Conditional Equity Premium in CESC.

A direct implication of the fact that the conditional invariant distribution of the welfare weights is non-degenerate is that the distribution of the conditional equity premium is non-degenerate as well. Figure 13 underscores that belief heterogeneity increases not only the support but also the volatility of the conditional equity premium. Most importantly, $E^{P_e}\left(R_{2,cpo}|\,\overline{R}_{1,cpo}\right)$ trends with probability close to one for moderately optimistic, correct and pessimistic beliefs and becomes larger as agent 2 becomes more pessimistic.

To understand why pessimism exacerbates trending, we approximate the conditional equity premium, $E^{P_e}\left(R_{2,cpo}|\,\overline{R}_{1,cpo}\right)(\omega)$, by

$$\pi^*\left(L|\,\xi_1\right)R^E_{cpo}(\xi_1,\alpha'_{cpo}\left(\xi_0,\alpha\right)(\xi_1))(1)+\pi^*\left(H|\,\xi_1\right)R^E_{cpo}(\xi_1,\alpha'_{cpo}\left(\xi_0,\alpha\right)(\xi_1))(2)\quad \text{if } \xi_1\in\{1,2\}$$

$$\pi^*\left(L|\,\xi_1\right)\;R^E_{cpo}(\xi_1,\alpha'_{cpo}\left(\xi_0,\alpha\right)(\xi_1))(3)+\pi^*\left(H|\,\xi_1\right)\;R^E_{cpo}(\xi_1,\alpha'_{cpo}\left(\xi_0,\alpha\right)(\xi_1))(4)\quad \text{if } \xi_1\in\{3,4\}$$

where $\pi^*\left(L|\,\xi_1\right)\equiv\left(\pi^*\left(1|\,\xi_1\right)+\pi^*\left(3|\,\xi_1\right)\right)$ and $\pi^*\left(H|\,\xi_1\right)\equiv\left(\pi^*\left(2|\,\xi_1\right)+\pi^*\left(4|\,\xi_1\right)\right)$. This approximation let us use the decomposition we introduce in (20) to explain trending.

To understand the impact of heterogeneous beliefs on the recession bias effect, note that for the recession states, $\xi'\in\{1,3\}$, the (inverse of) market pessimism is

$$\frac{m_{cpo}\left(\xi'\,|\xi,\alpha\right)}{\pi^*\left(\xi'\,|\xi\right)}=\begin{cases}\beta R^F_{cpo}\left(\xi,\alpha\right)\;g\left(L\right)^{-\sigma}\theta_{AJ}\left(\xi'\,|\xi,\alpha\right)\left(\frac{\pi_2\left(\xi'|\xi\right)}{\pi^*\left(\xi'|\xi\right)}\right)\theta_{BH}\left(\xi'\,|\xi,\alpha\right) & \text{if } \epsilon\left(\xi'\right)=\epsilon\left(\xi\right),\xi'\neq\xi\\ \beta R^F_{cpo}\left(\xi,\alpha\right)\;g\left(L\right)^{-\sigma}\theta_{AJ}\left(\xi'\,|\xi,\alpha\right) & \text{o.w.}\end{cases}$$

where $\theta_{AJ}\left(\xi'\,|\xi,\alpha\right)>1$ and $\theta_{BH}\left(\xi'\,|\xi,\alpha\right)<1$ are defined in Appendix C.3.

The market pessimism has four distinct ingredients: (1) the term $\theta_{AJ}\left(\xi'\,|\xi,\alpha\right)$ that corrects upwards the market belief of state ξ' to reflect the increase in the marginal valuation of consumption when, as in Alvarez and Jermann [3], the enforceability constraint binds due to a reversal in income shares; (2) the risk-free interest rate, $R^F_{cpo}\left(\xi',\alpha\right)$; (3) the ratio $\frac{\pi_2\left(\xi'|\xi\right)}{\pi^*\left(\xi'|\xi\right)}$ that adjusts the market belief to reflect the incorrect belief of agent 2; (4) the term $\theta_{BH}\left(\xi'\,|\xi,\alpha\right)$ that reflects the update in agent 2's welfare weight induced by belief heterogeneity. We underscore that agent 2's belief enters directly only in (3) and (4).

Figure 14 plots the distribution of the recession bias effect for some pessimistic beliefs of agent 2.

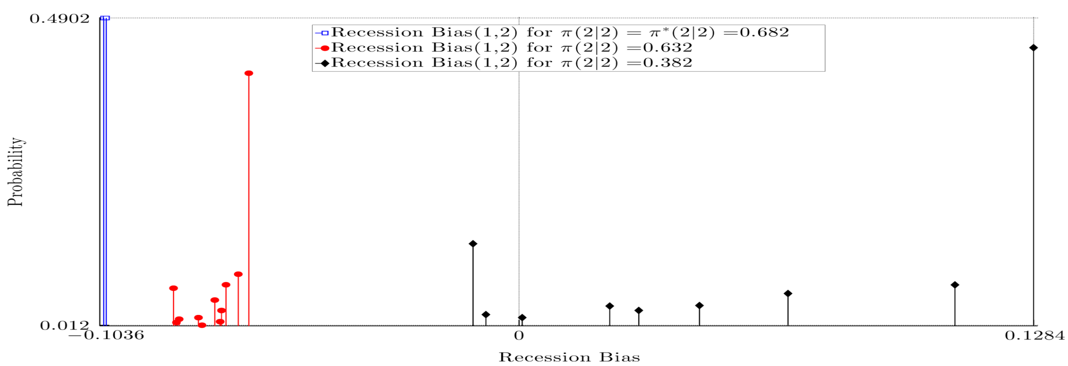

Figure 14: Distribution of the Recession Bias Effect in CESC.

When beliefs are homogeneous, the recession bias effect is negative. Agent 2's pessimism shifts the distribution of the recession bias effect to the right. For $\pi=0.382$ most of the mass is on the positive values and more than 45% is on the upper bound of the support. These changes can be explained in terms of the effect of agent 2's pessimism through the aforementioned ingredients (3) and (4).

First, since pessimism affects the persistency of expansions only, it means that both $\pi_2\left(1\,|2\right)$ and $\pi_2\left(3\,|4\right)$ increase while $\pi_2\left(1\,|1\right)$ and $\pi_2\left(3\,|3\right)$ remain constant. Consequently, the direct effect of pessimism is to increase the recession bias effect because only $\frac{m_{cpo}(1|2)}{\pi^*(1|2)}$ and $\frac{m_{cpo}(3|4)}{\pi^*(3|4)}$ are directly affected. Second, pessimism makes the welfare weights of agent 2 in states 1 and 3 to increase with respect to those in states 2 and 4, respectively. Consequently, pessimism induces a decrease in the marginal rate of substitution between any expansion state and the state next period with the same income share, decreasing the market belief about states 1 and 3 after states 2 and 4, respectively. This is captured by the new term $\theta_{BH}(\xi' | \xi, \alpha) < 1$. This second effect reduces the recession bias effect. Third, pessimism makes the interest rate to increase in expansions which makes the positive differential in the interest rates between recessions and expansions to shrink and, so, it increases the recession bias effect. When these three effect are added up, it turns out that the first and third ones dominate and this explains why the market belief about a recession, $m_{cpo}(1|\xi) + m_{cpo}(3|\xi)$, increases more at an expansion than at a recession as agent 2 becomes more pessimistic. Since the latter property implies that the ratio $\frac{\pi^*(1|\xi')+\pi^*(3|\xi')}{m_{cpo}(1|\xi)+m_{cpo}(3|\xi)}$ decreases more at an expansion than at a recession as agent 2 becomes more pessimistic, it explains why the market becomes more pessimistic at expansions than at recessions as the beliefs of agent 2 become pessimistic enough.

Now we turn to study the impact of return inertia effect. Figure 15 plots the distribution of the return inertia effect for different pessimistic beliefs of agent 2.

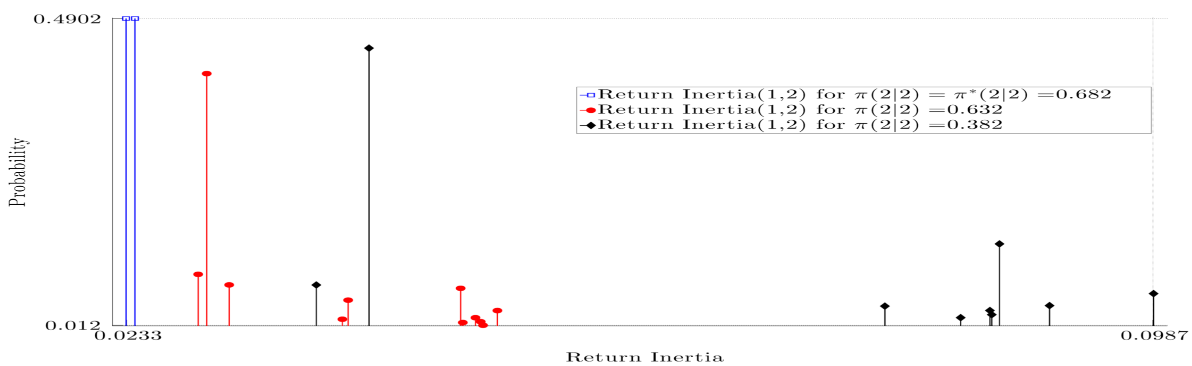

Figure 15: Distribution of the Return Inertia Effect in CESC.

When beliefs are homogeneous, the return inertia effect is degenerate, positive and small. As agent 2 becomes pessimistic, its distribution becomes nondegenerate and shifts to the right.

We conclude from Figures 14 and 15 that agent 2's pessimism makes both the recession bias and the return inertia effects stronger and that explains why the asset return displays larger trending than in the economy with homogeneous beliefs.

7 Our Model at Work: Final Discussion

In this section we present a final discussion about the role played by belief heterogeneity to explain short-term momentum and long-term reversal. The columns of Table 2 below summarise our main findings for short-term momentum (STM), long-term reversal (LTR), average equity premium ($E(R^E - R^F)$) and the average risk-free rate ($E(R^F)$). The β's are calibrated to match the historical average risk-free rate of 0.8% in the corresponding economy. The first row presents the U.S. data.

Table 2

	STM	LTR	$E(R^E - R^F)$	$E(R^F)$
U.S. Data (1900-2000)	13.9%	−15.3%	6.18%	0.8%
CE				
(i) $\sigma = 8.9$, $\beta = 1.109$	3%	−0.43%	2.57%	0.8%
(ii) $\sigma = 14.55$, $\beta = 1.107$	1.31%	−0.18%	6.18%	0.8%
CESC-AJ				
(i) $\sigma = 2.7$, $\beta = 0.860$	2.14%	−0.30%	1.02%	0.8%
(ii) $\sigma = 4.5$, $\beta = 0.648$	−0.07%	0.09%	6.24%	0.8%
CESC-HB				
(i) $\pi_2\left(2\vert\, 2\right) = 0.382$, $\sigma = 3$, $\beta = 0.756$	9.74%	−7.65%	6.18%	0.8%
(ii) $\pi_2\left(2\vert\, 2\right) = 0.222$, $\sigma = 2$, $\beta = 0.767$	14.18%	−8.37%	6.20%	0.8%

The second and third row report the prediction of CE for two cases. The first case uses the value of σ that delivers the largest STM. Since the signs of the autocorrelations are as in Table 1, the failure of the CE is only quantitative because it produces correlations of an order of magnitude lower than those observed in the data for STM (3% vs. 13.9%) and LTR (−0.43% vs. −15.3%). Not surprisingly, the CE equity premium (2.57%) falls short at explaining the average equity premium of 6.18%. The second case calibrates σ to match the average equity premium. In that case, the failure to explain STM and LTR is only quantitative but is exacerbated.

The fourth and fifth row report the prediction of the CESC with homogeneous correct beliefs (labelled CESC-AJ) for two cases where σ is chosen using the same criteria described above for CE. The row labeled CESC-AJ (i) shows the predictions when σ is set to deliver the largest value for STM. Once again, the failure of CESC with homogeneous beliefs is only quantitative since it produces autocorrelations with the correct signs but one order of magnitude lower than those observed in the data for STM (2.14% vs. 13.9%) and LTR (−0.30% vs. −15.3%). Moreover, the predicted equity premium is far from the target. In the row labelled CESC-AJ (ii), σ is set to match the average equity premium of 6.18%, the failure to explain these two anomalies is even starker than before since not even the signs are correct.

Finally, the sixth and seventh rows consider CESC models with heterogeneous beliefs (labelled CESC-HB) for two values of agent 2's pessimism. In CESC-HB (i), both σ and β are calibrated to match the equity premium and the risk-free rate when agent 2's pessimism is relatively moderate, i.e. $\pi_2\left(2\vert\, 2\right) = 0.382$ compared to $\pi^*\left(2\vert\, 2\right) = 0.682$. For a reasonable degree of relative risk aversion, $\sigma = 3$, the model predicts STM of 9.74% and LTR of −7.65%, a significant improvement with respect to CE (ii) and CESC-AJ (ii). In CESC-HB (ii) we make agent 2's pessimism stronger, $\pi_2\left(2\vert\, 2\right) = 0.222$. The model predicts STM of 14.18% and LTR of −8.37% while the corresponding calibrated sigma σ is quite moderate, i.e. $\sigma = 2$.

Appendix A

In this Appendix we prove the results in Section 3. We begin with some definitions.

Let $f : S \times \mathbb{R}_+^I \times \mathcal{P}(\Pi) \to \mathbb{R}_+$, $\|f\| \equiv \sup_{(\xi,\alpha,\mu)} \mid f(\xi,\alpha,\mu) : \alpha \in \Delta^{I-1} \mid$ and

$$\begin{aligned} F &\equiv \{f : S \times \mathbb{R}_+^I \times \mathcal{P}(\Pi) \to \mathbb{R}_+ :\ f \text{ is continuous and } \|f\| < \infty\}. \\ F_H &\equiv \left\{ f \in F : f(\xi,\alpha,\mu) - \sum_{i=1}^{I} \alpha_i U_i(\xi,\mu_i) \geq 0 \text{ for all } (\xi,\alpha,\mu), \text{ HOD 1 w.r.t. } \alpha \right\} \end{aligned}$$

where HOD 1 stands for *homogeneous of degree one*. F_H is a closed subset of the Banach space F and thus a Banach space itself. Continuity is with respect to the weak topology and thus the metric on F is induced by $\|.\|$.

The following Lemma characterises the utility possibility set and follows from a reasoning analogous to the one in Lemma 14 in Beker and Espino [5].

Lemma A.1. *$u \in \mathcal{U}^E(\xi,\mu)$ if and only if $u_i \geq U_i(\xi,\mu_i)$ for all i and*

$$\min_{\tilde{\alpha} \in \mathbb{R}_+^I} \left[v^*(\xi,\tilde{\alpha},\mu) - \sum_{i=1}^{I} \tilde{\alpha}_i\ u_i \right] \geq 0.$$

Given $(\xi,\alpha,\mu) \in S \times \mathbb{R}_+^I \times \mathcal{P}(\Pi)$, we define the operator T on F_H as follows

$$(Tf)(\xi,\alpha,\mu) = \max_{(c,w'(\xi'))} \sum_{i=1}^{I} \alpha_i \left\{ u_i(c_i) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \mid \xi)\ w_i'(\xi') \right\}, \tag{21}$$

subject to

$$c_i \geq 0, \qquad \sum_{i=1}^{I} c_i = y(\xi), \tag{22}$$

$$u_i(c_i) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \mid \xi)\ w_i'(\xi') \geq U_i(\xi,\mu_i), \tag{23}$$

$$w_i'(\xi') \geq U_i(\xi',\mu_i'(\xi,\mu)(\xi')) \qquad \text{for all } \xi', \tag{24}$$

$$\min_{\tilde{\alpha} \in \mathbb{R}_+^I} \left[f(\xi',\tilde{\alpha},\mu'(\xi,\mu)(\xi')) - \sum_{i=1}^{I} \tilde{\alpha}_i\ w_i'(\xi') \right] \geq 0 \qquad \text{for all } \xi' \tag{25}$$

and $\alpha'(\xi')$ solves (25).

Theorem 1 follows from Propositions A.2 and A.3. We say that $f \in F_H$ is *preserved under* T if $f(\xi,\alpha,\mu) \leq (Tf)(\xi,\alpha,\mu)$ for all (ξ,α,μ).

Proposition A.2. *If $f \in F_H$ is preserved under T, then $(Tf)(\xi,\alpha,\mu) \leq v^*(\xi,\alpha,\mu)$ for all (ξ,α,μ).*

Proof. Let $f \in F_H$ and define $\mathcal{W}(\xi,\mu)(f)$ as the constraint correspondence defined by (22)-(25) evaluated at f and (ξ,μ).

Let $\left(\tilde{c}_0, (\tilde{w}_1'(\xi'), \tilde{\alpha}_1'(\xi'))_{\xi'}\right) \in \mathcal{W}(\xi,\mu)(f)$. Since f is preserved under T, it follows from (25) that

$$\sum_{i=1}^{I} \tilde{\alpha}_{i,1}'(s_1)\tilde{w}_{i,1}'(s_1) \leq f(s_1,\tilde{\alpha}_1'(s_1),\mu_{s^1}) \leq (Tf)(s_1,\tilde{\alpha}_1'(s_1),\mu_{s^1}) \tag{26}$$

where the first inequality follows from (25) and the second one from the hypothesis. The second inequality implies that there exists some $\left(\tilde{c}_1\left(s_1\right), (\tilde{w}_2'(s_1,\xi'), \tilde{\alpha}_2'(s_1,\xi'))_{\xi'}\right) \in \mathcal{W}(s_1,\mu_{s^1})(f)$ such that

$$\tilde{w}_{i,1}'(s_1) = u_i(\tilde{c}_{i,1}\left(s_1\right)) + \beta(s_1,\mu_{s^1}) \sum_{\xi'} \pi_{\mu_{i,s^1}}(\xi' \left| s_1 \right.) \ \tilde{w}_{i,2}'(s_1,\xi') \text{ for all } i$$

Following this strategy, one can construct a collection of functions $\{\tilde{c}_t\left(s^t\right), \tilde{w}_t\left(s^t\right)\}$ for all s^t and $t \geq 1$. Define, $\{c_t\}_{t=0}^{\infty} \in \mathbb{C}(\xi)$ as follows:

$$\begin{aligned} c_0 = \tilde{c}_0, \quad c_t\left(s\right) = \tilde{c}_t\left(s^t\right) & \qquad \text{for all } s \text{ and } t \geq 1, \\ w_t\left(s\right) = \tilde{w}_t\left(s^t\right) & \qquad \text{for all } s \text{ and } t \geq 1. \end{aligned}$$

Since $\{c_t\}_{t=0}^{\infty}$ is feasible by construction, we show it is enforceable. By construction, we have that

$$\begin{aligned} \left|U_i(c_i)(s^t) - \tilde{w}_{i,t}'(s^t)\right| &\leq \overline{\beta}\left|\sum_{\xi'} \pi_{\mu_{i,s^t}}(\xi' \left| s_t \right.) \left(U_i(c_i)(s^t,\xi') - \tilde{w}_{i,t}'(s^t,\xi')\right)\right| \\ &\leq \overline{\beta}\sup_{\xi'}\left|U_i(c_i)(s^t,\xi') - \tilde{w}_{i,t}'(s^t,\xi')\right| \\ &\leq \overline{\beta}^k \sup_{(\xi_1',...\xi_k')}\left|U_i(c_i)(s^t,\xi_1',...\xi_k') - \tilde{w}_{i,t}'(s^t,\xi_1',...\xi_k')\right|. \end{aligned}$$

Since $U_i(s_t,\mu_{i,s^t}) \leq \tilde{w}_{i,t}' \leq \|f\| < \infty$ for all i and t and $U_i\left(\cdot\right)$ is uniformly bounded, it follows that

$$\begin{aligned} \left|U_i(c_i)(s^t) - \tilde{w}_{i,t}'(s^t)\right| &\leq \limsup_{k\to\infty}\left\{\overline{\beta}^k \sup_{(\xi_1',...\xi_k')}\left|U_i(c_i)(s^t,\xi_1',...\xi_k') - \tilde{w}_{i,t}'(s^t,\xi_1',...\xi_k')\right|\right\} \\ &= 0. \end{aligned}$$

and consequently $U_i(c_i)(s^t) = \tilde{w}_{i,t}'(s^t)$ for all i and all s^t. Finally, since by construction $\tilde{w}_{i,t}'(s^t) \geq U_i(s_t,\mu_{i,s^t})$ for all i, we can conclude that $\{c_t\}_{t=0}^{\infty}$ is enforceable.

We conclude that for any arbitrary $\alpha \in \mathbb{R}_+^I$

$$\begin{aligned} & \sum_{i=1}^{I}\alpha_i\left(u_i(\tilde{c}_{i,0}) + \beta(\xi,\mu_i)\sum_{\xi'}\pi_{\mu_i}(\xi' \left| \xi \right.) \ \tilde{w}_{i,1}'(\xi')\right) \\ = \ & \sum_{i=1}^{I}\alpha_i u_i(c_{i,0}) + \sum_{i=1}^{I}\alpha_i E^{P_i}\left(\rho_{i,0} \ w_{i,1}'\right) \\ = \ & \sum_{i=1}^{I}\alpha_i E^{P_i}\left(\sum_{t=0}^{T}\rho_{i,t} \ u_i(c_{i,t})\right) + \sum_{i=1}^{I}\alpha_i E^{P_i}\left(\rho_{i,T+1} \ w_{i,T+1}'\right) \\ \leq \ & \sum_{i=1}^{I}\alpha_i E^{P_i}\left(\sum_{t=0}^{T}\rho_{i,t} \ u_i(c_{i,t})\right) + \overline{\beta}^{T+1}\left\|f\right\|. \end{aligned}$$

where the inequality follows from the first inequality in (26). Taking limits, we obtain

$$\sum_{i=1}^{I}\alpha_i\left(u_i(\tilde{c}_{i,0}) + \beta(\xi,\mu_i)\sum_{\xi'}\pi_{\mu_i}(\xi' \left| \xi \right.) \ \tilde{w}_{i,1}'(\xi')\right) \leq \sum_{i=1}^{I}\alpha_i E^{P_i}\left(\sum_{t=0}^{\infty}\rho_{i,t} \ u_i(c_{i,t})\right) \leq v^*(\xi,\alpha,\mu).$$

where the first inequality follows because weak inequalities are preserved under limits and $\overline{\beta} \in (0,1)$ and the last one because $\{c_t\}_{t=0}^{\infty}$ is enforceable.

Since $\left(\tilde{c}_0, (\tilde{w}_1'(\xi'), \tilde{\alpha}_1'(\xi'))_{\xi'}\right) \in \mathcal{W}(\xi,\mu)(f)$ is arbitrary,

$$\sum_{i=1}^{I} \alpha_i \left(u_i(\tilde{c}_{i,0}) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_\mu(\xi' \,|\xi\,)\ \tilde{w}_{i,1}'(\xi') \right) \leq v^*(\xi,\alpha,\mu),$$

for all $\left(\tilde{c}_0, (\tilde{w}_1'(\xi'), \tilde{\alpha}_1'(\xi'))_{\xi'}\right) \in \mathcal{W}(\xi,\mu)(f)$. Therefore,

$$\begin{aligned} Tf(\xi,\alpha,\mu) &= \max_{(c,w')\in\mathcal{W}(\xi,\mu)} \sum_{i=1}^{I} \alpha_i \left(u_i(c_i) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_\mu(\xi' \,|\xi\,)\ w_i'(\xi') \right) \\ &\leq v^*(\xi,\alpha,\mu). \end{aligned}$$

as desired. □

Proposition A.3. *$v^* \in F_H$ is preserved under T and $v^*(\xi,\alpha,\mu) = (Tv^*)(\xi,\alpha,\mu)$ for all (ξ,α,μ).*

Proof. Given (ξ,α,μ), take any $u \in \mathcal{U}^E(\xi,\mu)$ and let $c \in \mathbb{C}(\xi)$ denote the corresponding enforceable feasible allocation. For each ξ', ${}_{\xi'}c_i \in \mathbb{C}(\xi')$ given by

$${}_{\xi'}c_i(s^t) = c_i(\xi', s^t) \text{ for all } t \geq 1,$$

denotes the $\xi'-$continuation of c_i. For every $t \geq 1$, let

$$P_{i,\xi'}(s^t) = \frac{P_i(C(s^t))}{\pi_{\mu_{i,s^t}}(\xi' \,|s_t\,)},$$

and note that

$$\sum_{i=1}^{I} \alpha_i U_i^{P_i}(c_i) = \sum_{i=1}^{I} \alpha_i \left[u_i(c_{i,0}) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi\,)) U_i^{P_{i,\xi'}}({}_{\xi'}c_i) \right].$$

Since $\left(U_i^{P_{i,\xi'}}({}_{\xi'}c_i)\right)_{i=1}^{I} \in \mathcal{U}(\xi', \mu'(\xi,\mu)(\xi'))$ for all ξ', it follows by Lemma A.1 that

$$\sum_{i=1}^{I} \alpha_i' U_i^{P_{i,\xi'}}({}_{\xi'}c_i) \leq v^*(\xi', \mu'(\xi,\mu)(\xi'), \alpha') \quad \text{for all } \xi' \text{ and } \alpha' \in \Delta^{I-1}$$

and so

$$\sum_{i=1}^{I} \alpha_i U_i^{P_i}(c_i) \leq (Tv^*)(\xi,\alpha,\mu) \quad \text{for all } \xi' \text{ and } \alpha' \in \Delta^{I-1}.$$

We conclude that v^* is preserved under T since

$$v^*(\xi,\alpha,\mu) = \sup_{c\in Y^\infty} \sum_{i=1}^{I} \alpha_i U_i^{P_i}(c_i) \leq (Tv^*)(\xi,\alpha,\mu) \quad \text{for all } (\xi,\alpha,\mu).$$

It follows from Proposition A.2 that $(Tv^*)(\xi,\alpha,\mu) \leq v^*(\xi,\alpha,\mu)$ and so $v^*(\xi,\alpha,\mu) = (Tv^*)(\xi,\alpha,\mu)$ for all (ξ,α,μ). □

Proof of Theorem 1. Since Proposition A.3 shows that v^* is a fixed point of T, the rest of the proof is analogous to that of Theorem 2 in Beker and Espino [5]. □

Proof of Proposition 3. Note that T^n is a monotone operator for all $n \geq 1$ (i.e., if $f \geq g$ then $T^n f \geq T^n g$.) Let $\tilde{T}$ be the operator when the enforceability constraints are ignored. Theorem 2 in Beker and Espino [5] show that $\tilde{T}$ has a unique fixed point, say $\tilde{v}$.

Let $\{v_n\}_{n=0}^{\infty}$ be the sequence of functions defined by $v_0 = \tilde{v}$ and $v_n = T(v_{n-1})$ for all $n \geq 1$. Next we show that $v_n \geq v_{n+1} \geq v^*$ for all n. Indeed, since $T(\tilde{v}) \leq \tilde{T}(\tilde{v}) = \tilde{v}$, it follows that $v_1 \leq v_0$ and

$$\begin{aligned} v_{n+1} &= T^{n+1}(\tilde{v}) = T^n(T\tilde{v}) = T^n(v_1) \leq T^n(v_0) = v_n \quad \text{for all } n, \\ v_{n+1} &= T^{n+1}(\tilde{v}) \geq T^{n+1}(v^*) = v^* \quad \text{for all } n. \end{aligned}$$

Since $\{v_n\}_{n=0}^{\infty}$ is a monotone decreasing sequence of uniformly bounded functions bounded below by v^*, there exists a uniformly bounded function $v_\infty \geq v^*$ such that $\lim_{n\to\infty} v_n = v_\infty$. To show that $v_\infty \leq v^*$ we argue that v_∞ is preserved under T and apply Proposition A.2.

Given (ξ, α, μ), $v_\infty(\xi, \alpha, \mu) \leq v_n(\xi, \alpha, \mu)$ for all n and so there is $\left(\tilde{c}_n, (\tilde{w}'_n(\xi'), \tilde{\alpha}'_n(\xi'))_{\xi'}\right) \in W(\xi, \mu)(v_n)$ such that

$$v_n(\xi, \alpha, \mu) = \sum_{i=1}^{I} \alpha_i \left(u_i(\tilde{c}_{i,n}) + \beta(\xi, \mu_i) \sum_{\xi'} \pi_\mu(\xi' \,|\, \xi)\ \tilde{w}'_{i,n}(\xi') \right), \quad \text{for all } n. \tag{27}$$

Since $\left(\tilde{c}_n, (\tilde{w}'_n(\xi'), \tilde{\alpha}'_n(\xi'))_{\xi'}\right)$ lies in a compact set, it has a convergent subsequence with limit point $\left(\tilde{c}, (\tilde{w}'(\xi'), \tilde{\alpha}'(\xi'))_{\xi'}\right)$.

Note that

$$\begin{aligned} v_n(\xi', \alpha'(\xi, \mu)(\xi'), \mu'(\xi, \mu)(\xi')) - \sum_{i=1}^{I} \alpha'_i\ \tilde{w}'_{i,n}(\xi') &\geq 0, \\ \tilde{w}'_{i,n}(\xi') - U_i(\xi, \mu_i) &\geq 0, \end{aligned}$$

for all n and all ξ'. Since weak inequalities are preserved in the limit

$$\begin{aligned} v_\infty(\xi', \alpha'(\xi, \mu)(\xi'), \mu'(\xi, \mu)(\xi')) - \sum_{i=1}^{I} \alpha'_i\ \tilde{w}'_i(\xi') &\geq 0, \\ \tilde{w}'_i(\xi') - U_i(\xi, \mu_i) &\geq 0, \end{aligned}$$

and, therefore, $\left(\tilde{c}, (\tilde{w}'(\xi'), \tilde{\alpha}'(\xi'))_{\xi'}\right) \in W(\xi, \mu)(v_\infty)$. Consequently,

$$\begin{aligned} (Tv_\infty)(\xi, \alpha, \mu) &\geq \sum_{i=1}^{I} \alpha_i \left(u_i(\tilde{c}_i(\xi)) + \beta(\xi, \mu_i) \sum_{\xi_2} \pi_{\mu_i}(\xi' \,|\, \xi)\ \tilde{w}'_i(\xi') \right) \\ &= v_\infty(\xi, \alpha, \mu). \end{aligned}$$

where the equality follows by (27) and continuity. □

The following Lemma will be used in the proof of Proposition 4

Lemma A.4. *If A1 and A3 holds, there exists N and α^* such that $(\xi_N(\omega), \alpha_N(\omega)) = (\xi^*, \alpha^*)$ for all $\omega \in \{\tilde{\omega} : \xi_t(\tilde{\omega}) = \xi^*$ for t even, $\xi_t(\tilde{\omega}) = \xi^{**}$ for t odd, $1 \leq t \leq N\}$.*

Proof. Let $\delta \equiv \frac{\pi_1(\xi^*|\xi^{**})}{\pi_2(\xi^*|\xi^{**})}\frac{\pi_1(\xi^{**}|\xi^*)}{\pi_2(\xi^{**}|\xi^*)}$. Consider the case in which $\delta > 1$. Note that without loss of generality we can assume $\frac{\pi_1(\xi^{**}|\xi^*)}{\pi_2(\xi^{**}|\xi^*)} > 1$. Let $\alpha_1^* \equiv \max_{\alpha \in \Delta(\xi^{**},\mu^\pi)} \alpha'_{1,cpo}(\xi^{**},\alpha,\mu^\pi)(\xi^*)$. Let N^* be the smallest $n \in \mathbb{N} \cup \{0\}$ satisfying

$$\delta^n \frac{\underline{\alpha}_1(\xi^*,\mu^\pi)}{1-\underline{\alpha}_1(\xi^*,\mu^\pi)} \geq \frac{\alpha_1^*}{1-\alpha_1^*} > \delta^{n-1}\frac{\underline{\alpha}_1(\xi^*,\mu^\pi)}{1-\underline{\alpha}_1(\xi^*,\mu^\pi)}.$$

For any $\alpha \in \Delta(\xi^*,\mu^\pi)$ such that $\alpha_1 \leq \alpha_1^*$

$$\frac{\alpha'_{1,cpo}(\xi^*,\alpha,\mu^\pi)(\xi^{**})}{1-\alpha'_{1,cpo}(\xi^*,\alpha,\mu^\pi)(\xi^{**})} = \max\left\{\frac{\alpha'_{1,po}(\xi^*,\alpha,\mu^\pi)(\xi^{**})}{1-\alpha'_{1,po}(\xi^*,\alpha,\mu^\pi)(\xi^{**})}, \frac{\underline{\alpha}_1(\xi^{**},\mu^\pi)}{1-\underline{\alpha}_1(\xi^{**},\mu^\pi)}\right\}$$

and so $\alpha'_{1,cpo}(\xi^*,\alpha,\mu^\pi)(\xi^{**}) \geq \alpha'_{1,po}(\xi^*,\alpha,\mu^\pi)(\xi^{**})$. It follows that

$$\begin{aligned}
\frac{\alpha'_{1,cpo}(\xi^{**},\alpha'_{cpo}(\xi^*,\alpha,\mu^\pi)(\xi^{**}),\mu^\pi)(\xi^*)}{1-\alpha'_{1,cpo}(\xi^{**},\alpha'_{cpo}(\xi^*,\alpha,\mu^\pi)(\xi^{**}),\mu^\pi)(\xi^*)} &\geq \frac{\alpha'_{1,cpo}(\xi^{**},\alpha'_{po}(\xi^*,\alpha,\mu^\pi)(\xi^{**}),\mu^\pi)(\xi^*)}{1-\alpha'_{1,cpo}(\xi^{**},\alpha'_{po}(\xi^*,\alpha,\mu^\pi)(\xi^{**}),\mu^\pi)(\xi^*)} \\
&\geq \frac{\pi_1(\xi^*|\xi^{**})}{\pi_2(\xi^*|\xi^{**})}\frac{\alpha'_{1,po}(\xi^*,\alpha,\mu^\pi)(\xi^{**})}{1-\alpha'_{1,po}(\xi^*,\alpha,\mu^\pi)(\xi^{**})} \\
&= \frac{\pi_1(\xi^*|\xi^{**})}{\pi_2(\xi^*|\xi^{**})}\frac{\pi_1(\xi^{**}|\xi^*)}{\pi_2(\xi^{**}|\xi^*)}\frac{\alpha_1}{1-\alpha_1} \\
&= \delta\frac{\alpha_1}{1-\alpha_1}.
\end{aligned}$$

Let $N \equiv 2(N^*+1)$. Consider $\omega \in \Omega^* \equiv \{\tilde{\omega} : \xi_t(\tilde{\omega}) = \xi^*$ for t even, $\xi_t(\tilde{\omega}) = \xi^{**}$ for t odd, $1 \leq t \leq N\}$. The sequence $\{\alpha_{1,t}(\omega)\}$ generated by α'_{cpo} satisfies $\underline{\alpha}_1(\xi^*,\mu^\pi) \leq \alpha_{1,t}(\omega) \leq \alpha_1^*$ and, therefore, $\alpha_{1,t+2}(\omega) \geq \delta\alpha_{1,t}(\omega)$ for all even t such that $2 \leq t \leq N$. Thus, for any even t such that $2 \leq t \leq N$

$$\alpha_{1,t}(\omega) \geq \min\left\{\delta^{\frac{t-2}{2}}\frac{\alpha_{1,2}(\omega)}{1-\alpha_{1,2}(\omega)}, \frac{\alpha_1^*}{1-\alpha_1^*}\right\}$$

and so it follows by the definition of N^* that $\alpha_N(\omega) = \alpha^*$.

If $\delta < 1$, we define $\alpha_1^* \equiv \min_{\alpha \in \Delta(\xi^{**},\mu^\pi)} \alpha'_{1,cpo}(\xi^{**},\alpha,\mu^\pi)(\xi^*)$ and the proof is analogous to the case $\delta > 1$. $\square$

Proof of Proposition 4. The existence of a unique invariant distribution that is globally stable follows by Theorem 11.12 in Stokey and Lucas [33]. It suffices to show that F_{cpo} satisfies the following condition:

<u>Condition M</u>: There exists $\epsilon > 0$ and an integer $N \geq 1$ such that for any $A \in \mathscr{S}$, either $P^N(s,A) \geq \epsilon$, all $s \in S$, or $P^N(s,A^c) \geq \epsilon$, all $s \in S$.

Define N and α^* as in Lemma A.4 and $\epsilon \equiv (\min_{\xi} \pi^*(\xi^{**}|\xi))\big(\pi^*(\xi^*|\xi^{**})\pi^*(\xi^{**}|\xi^*)\big)^N > 0$. Let $A \in \mathscr{S}$ and $(\xi,\alpha) \in \Omega$ be arbitrary. If $\alpha^* \in A$, then $P^N((\xi,\alpha),A) \geq P^N((\xi,\alpha),\alpha^*) \geq \epsilon$ by Lemma A.4. If $\alpha^* \in A^c$, then $P^N((\xi,\alpha),A^c) \geq P^N((\xi,\alpha),\alpha^*) \geq \epsilon$ by Lemma A.4 . To show the invariant distribution is not degenerate note that α_1^* must be part of the support. If $\alpha_1^* \notin \Delta(\xi^{**},\mu^\pi)$, the result follows trivially. If $\alpha_1^* \in \Delta(\xi^{**},\mu^\pi)$, either $\alpha'_{1,cpo}(\xi^{**},\alpha^*,\mu^\pi)(\xi^*) \neq \alpha^*$ or $\alpha'_{1,cpo}(\xi^*,\alpha^*,\mu^\pi)(\xi^{**}) \neq \alpha^*$. $\square$

Proof of Proposition 5. Since risk-free rates are assumed to be positive and (ξ,α,μ) lies in a compact set, it follows by continuity of R^F that $R^F_{\min} \equiv \min_{(\xi,\alpha,\mu)} R^F(\xi,\alpha,\mu) > 1$. Let $m(\xi' | \xi,\alpha,\mu) \equiv Q(\xi,\alpha,\mu)(\xi')R^F(\xi,\alpha,\mu)$ be the equivalent martingale measure.

Let $f \in F$ and consider the operator T_i defined by

$$(T_i f)(\xi,\alpha,\mu) = c_i(\xi,\alpha) - y_i(\xi) + \frac{\sum_{\xi'} m(\xi' | \xi,\alpha,\mu) f(\xi', \alpha'_{cpo}(\xi,\alpha,\mu)(\xi'), \mu'(\xi,\mu)(\xi'))}{R^F(\xi,\alpha,\mu)}.$$

Step 1: We check that $T_i : F \to F$.

Suppose that $f \in F$. Since α' and μ' are both continuous, then

$$\frac{\sum_{\xi'} m(\xi' | \xi,\alpha,\mu) f(\xi', \alpha'_{cpo}(\xi,\alpha,\mu)(\xi'), \mu'(\xi,\mu))}{R^F(\xi,\alpha,\mu)}, \tag{28}$$

is continuous in (ξ,α,μ). Also, (28) is bounded because f and R^F are both bounded. Since $|c_i(\xi,\alpha) - y_i(\xi)|$ is uniformly bounded, we can conclude that $(T_i f) \in F$.

Step 2: We check that T_i satisfies Blackwell's sufficient conditions for a contraction mapping.

Discounting. Consider any $a > 0$ and note that

$$\begin{aligned} T_i(f+a)(\xi,\alpha,\mu) &= c_i(\xi,\alpha) - y_i(\xi) + \frac{\sum_{\xi'} m(\xi' | \xi,\alpha,\mu) f(\xi', \alpha'_{cpo}(\xi,\alpha,\mu)(\xi'), \mu'(\xi,\mu)(\xi'))}{R^F(\xi,\alpha,\mu)} + \frac{a}{R^F(\xi,\alpha,\mu)} \\ &\leq (T_i f)(\xi,\alpha,\mu) + \left(R^F_{\min}\right)^{-1} a. \end{aligned}$$

Monotonicity. If $f(\xi,\alpha,\mu) \geq g(\xi,\alpha,\mu)$ for all (ξ,α,μ), it is immediate that $(T_i f)(\xi,\alpha,\mu) \geq (T_i g)(\xi,\alpha,\mu)$ for all (ξ,α,μ).

Thus, we can apply the Contraction Mapping Theorem to conclude that T_i is a contraction with a unique fixed point $A_i \in F$ and that the fixed point is the unique solution to (11) for each i. Finally, the same arguments used in Espino and Hintermaier [17] show that there exists $\alpha_0 = \alpha(s_0,\mu_0) \in \mathbb{R}^I_+$ such that $A_i(s_0,\alpha_0,\mu_0) = 0$ for all i. $\square$

Proof of Theorem 6. Given q and B_i, we argue that (c_i, a_i) satisfies *(CESC 1)*.

First, we argue that (c_i, a_i) is in agent i's budget set. Note that the solvency constrains are satisfied by construction. Since $a^{s_0}_{i,-1} = 0$ for all i, it follows by construction of (c_i, a_i) and the definition of A_i that the sequential budget constraint is satisfied.

Next, we argue that (c_i, a_i) is optimal given q and B_i. Notice that (13) implies that

$$\begin{aligned} q^{\xi'}_t(s) &= \max_h \left\{ \beta(s_t, \mu_{h,s^t}) \ \pi_{\mu_{h,s^t}}(\xi' | s_t) \frac{u'_h(c_{h,t+1}(s'))}{u'_h(c_{h,t}(s))} \right\} \text{ where } s' \in C\left(s^t, \xi'\right), \\ &\geq \beta(s_t, \mu_{i,s^t}) \, \pi_{\mu_{i,s^t}}(\xi' | s_t) \frac{u'_i(c_{i,t+1}(s'))}{u'_i(c_{i,t}(s))}. \end{aligned} \tag{29}$$

for all i (with equality if $U_i(c_i)(s^t) > U_i(s_t, \mu_{i,s^{t-1}})$). Consider any alternative plan $(\tilde{c}_i, \tilde{a}_i)$ in agent i's budget set. It follows by concavity that

$$u_i(c_{i,t}) - u_i(\tilde{c}_{i,t}) \geq u'_i(c_{i,t})\,(c_{i,t} - \tilde{c}_{i,t}) \tag{30}$$

while

$$c_{i,t}(s) - \tilde{c}_{i,t}(s) = a^{s_t}_{i,t-1}(s) - \tilde{a}^{s_t}_{i,t-1}(s) + \sum_{\xi'} q^{\xi'}_t(s) \left(\tilde{a}^{\xi'}_{i,t}(s) - a^{\xi'}_{i,t}(s)\right) = -b_{i,t}(s) + b^*_{i,t}(s),$$

where $b_{i,0} \equiv 0$ and

$$b_{i,t}(s) \equiv \tilde{a}^{s_t}_{i,t-1}(s) - a^{s_t}_{i,t-1}(s) = \tilde{a}^{s_t}_{i,t-1}(s) - A(s_t, \alpha_t(s), \mu_{s^t}) = \tilde{a}^{s_t}_{i,t-1}(s) - B^{s_t}_{i,t-1}(s) \quad \text{for } t \geq 1,$$

$$b^*_{i,t}(s) \equiv \sum_{\xi'} q^{\xi'}_t(s)\left(\tilde{a}^{\xi'}_{i,t}(s) - a^{\xi'}_{i,t}(s)\right) \quad \text{for } t \geq 0.$$

Note that

$$\begin{aligned} u'_i(c_{i,t}(s))b^*_{i,t} &= u'_i(c_{i,t}(s))\sum_{\xi'} q^{\xi'}_t(s)\left(\tilde{a}^{\xi'}_{i,t}(s) - a^{\xi'}_{i,t}(s)\right) \\ &= \sum_{\xi'} u'_i(c_{i,t}(s))q^{\xi'}_t(s)\left(\tilde{a}^{\xi'}_{i,t}(s) - a^{\xi'}_{i,t}(s)\right) \\ &\geq E^{P_i}\left[\beta u'_i(c_{i,t+1})b_{i,t+1} \middle| \mathcal{F}_t\right]. \end{aligned} \tag{31}$$

where the inequality follows from (29). For $T < \infty$, let $\Delta \equiv E^{P_i}\left(\sum_{t=0}^{T} \rho_t \ (u_i(c_{i,t}) - u_i(\tilde{c}_{i,t}))\right)$. Then,

$$\begin{aligned} \Delta &\geq E^{P_i}\left(\sum_{t=0}^{T} \rho_t u'_i(c_{i,t})\left(c_{i,t} - \tilde{c}_{i,t}\right)\right) \\ &= E^{P_i}\left(\sum_{t=0}^{T} \rho_t u'_i(c_{i,t})\left(-b_{i,t} + b^*_{i,t}\right)\right) \\ &= -E^{P_i}\left[\sum_{t=0}^{T} \rho_t u'_i(c_{i,t})b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} \rho_t u'_i(c_{i,t})b^*_{i,t}\right] \\ &= -E^{P_i}\left[\sum_{t=0}^{T} \rho_t u'_i(c_{i,t})b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} \rho_t E^{P_i}\left[u'_i(c_{i,t})b^*_{i,t} \middle| \mathcal{F}_t\right]\right] \\ &\geq -E^{P_i}\left[\sum_{t=0}^{T} \rho_t u'_i(c_{i,t})b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} E^{P_i}\left[\rho_{t+1} \ u'_i(c_{i,t+1})b_{i,t+1} \middle| \mathcal{F}_t\right]\right] \\ &= -E^{P_i}\left[\sum_{t=0}^{T} \rho_t u'_i(c_{i,t})b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} \rho_{t+1} u'_i(c_{i,t+1})b_{i,t+1}\right], \end{aligned}$$

where the first line uses (30), the fourth and last lines follows from the law of iterated expectations and the inequality in the fifth line follows from (31). Since $b_{i,0} = 0$,

$$\Delta \equiv E^{P_i}\left[\sum_{t=0}^{T} \rho_t \ (u_i(c_{i,t}) - u_i(\tilde{c}_{i,t}))\right] \geq E^{P_i}\left[\rho_{T+1} u'_i(c_{i,T+1}(s))b_{i,T+1}\right].$$

Now we argue that $b_{i,t}$ is uniformly bounded. Since $\mathcal{P}(\Pi^K)$ is compact (in the weak topology), the continuous functions $\underline{\alpha}_i(\xi, \mu)$ and $A_i(\xi, \alpha, \mu)$ and, therefore, $B^{\xi'}_{i,t}$ is uniformly bounded for all ξ'. So it suffices to show $\tilde{a}^{\xi'}_{i,t}$ is uniformly bounded for all ξ'. Note that $\tilde{a}^{\xi'}_{i,t}(s)$ is bounded below by $B^{\xi'}_{i,t}(s)$ and market clearing implies it is bounded above by $-B^{\xi'}_{j,t}(s)$ for $j \neq i$.

It follows from the Dominated Convergence Theorem that

$$\begin{aligned} E^{P_i}\left[\sum_{t=0}^{\infty} \rho_t \ (u_i(c_{i,t}) - u_i(\tilde{c}_{i,t}))\right] &= \lim_{T\to\infty} E^{P_i}\left[\sum_{t=0}^{T} \rho_t \ (u_i(c_{i,t}) - u_i(\tilde{c}_{i,t}))\right] \\ &\geq \lim_{T\to\infty} E^{P_i}\left[\rho_{T+1} u'_i(c_{i,T+1})b_{i,T+1}\right] \\ &= 0. \end{aligned}$$

since $\beta(\xi, \mu) \leq \overline{\beta} \in (0, 1)$ for all (ξ, μ). Consequently, given q and B_i, (c_i, a_i) solves agent i's problem.

Finally, note that $\{a_i\}$ satisfies *(CESC 2)* since $\sum_{i=1}^{I} A_i(\xi, \alpha, \mu) = 0$ for all (ξ, α, μ) (see (12)). $\square$

Appendix B

In this Appendix we prove the results of Section 5.

Theorem B.1 (Stout [34] and Jensen and Rahbek [19]). *Assume* $\{z_t\}_{t=0}^{\infty}$ *is a time homogeneous Markov process with transition function* F *on* $(Z, \mathcal{Z})$. *If there exists a unique invariant distribution* $\psi : \mathcal{Z} \to [0,1]$, *then for any* $z_0 \in Z$, *any integer* k *and any continuous function* $f : Z^k \to \Re$,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} f\left(z_t, ..., z_{t+k}\right) = E^{P^F(\psi,\cdot)}\left(f\left(\tilde{z}_0, ..., \tilde{z}_k\right)\right), \qquad P^F\left(z_0, \cdot\right) - a.s.$$

Proof of Theorem 8. For the case of CPO allocations when agents have dogmatic priors, the result follows directly from Proposition 4. So we only deal here with the case of PO. Under our assumptions *PO* allocations can be parameterized by welfare weights. Let agent h be some agent whose prior satisfies A2. A straightforward extension of Beker and Espino [5] to handle Markov uncertainty can be used to show that the welfare weights associated with a *PO* allocation satisfy that for every agent i and every path $s \in S^{\infty}$

$$\alpha_{i,t}\left(s\right) = \frac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\sum_{j=1}^{I} \alpha_{j,0}\ P_{j,t}\left(s\right)} = \frac{\dfrac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)}}{\sum\limits_{i=1}^{I} \dfrac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)}}$$

and so the limit behaviour of the welfare weights depends on the limit behaviour of the likelihood ratio

$$\frac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)}.$$

If h's prior satisfies A1.a then one can use Sandroni's results to show that, $P^{\pi^*} - a.s.$,

$$\frac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)} \to \frac{\alpha_{i,0}}{\alpha_{h,0}} \frac{\mu_i\left(\pi^*\right)}{\mu_h\left(\pi^*\right)}$$

while if h's prior satisfies A1.b then one can use Phillip and Ploberger's [28, Theorem 4.1] results to show that, $P^{\pi^*} - a.s.$,

$$\frac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)} \to \frac{\alpha_{i,0}}{\alpha_{h,0}} \frac{f_i\left(\pi^*\right)}{f_h\left(\pi^*\right)}.$$

It follows that $\alpha_{i,t}\left(s\right) \to \alpha_{\infty}$, $P^{\pi^*} - a.s.$

Since every agent's prior satisfies A1, it is well known that there exists some $\pi = (\pi_1, ..., \pi_I)$ where $\pi_i \in \Pi^K$ such that μ_{i,s^t} converges weakly to μ^{π_i} for $P^{\pi_i}-$almost all $s \in S^{\infty}$ and π_i is the element of i's support which is closer to π^* in terms of entropy. By assumption A.2, $\pi_h = \pi^*$.

Since convergence almost surely implies convergence in distribution, we conclude that, $P^{\pi^*} - a.s.$, the marginal distribution over welfare weights and beliefs converges to a mass point on $(\alpha_{\infty}, \mu^{\pi})$. □

Proof of Proposition 9. We need to show that

$$E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right](\omega) > 0 \qquad \text{if } E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right) \text{ trends,} \tag{32}$$

$$E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right](\omega) < 0 \qquad \text{if } E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right) \text{ reverts to the mean.} \tag{33}$$

Let Ω_e be the support of P_e, $\Omega^+ \equiv \left\{\tilde{\omega} \in \Omega_e : \overline{R}_{1,e}(\tilde{\omega}) \geq 0\right\}$ and $\Omega^- \equiv \left\{\tilde{\omega} \in \Omega_e : \overline{R}_{1,e}(\tilde{\omega}) < 0\right\}$. Note that

$$\begin{aligned} E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right](\omega) &= P_e\left(\Omega^+\right)\ E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\middle|\Omega^+\right] + \\ &\quad P_e\left(\Omega^-\right)\ E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\middle|\Omega^-\right] \end{aligned}$$

and so $E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right](\omega)$ is bounded below by

$$\begin{aligned} &P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^+\right)\inf_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) + \\ &P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^-\right)\sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \end{aligned}$$

and above by

$$\begin{aligned} &P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^+\right)\sup_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) + \\ &P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^-\right)\inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \end{aligned}$$

If $E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\omega)$ trends, then

$$\inf_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) > \sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega})$$

and so (32) holds because

$$\begin{aligned} E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right](\omega) &> \left(P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^+\right)\right)\sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) + \\ &\quad \left(P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^-\right)\right)\sup_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \\ &= E^{P_e}\left(\overline{R}_{1,e}\right)\sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \\ &= 0. \end{aligned}$$

If $E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\omega)$ reverts to the mean, then

$$\sup_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) < \inf_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\omega)$$

and so (33) holds because

$$\begin{aligned} E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)\right](\omega) &< \left(P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^+\right)\right)\inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) + \\ &\quad \left(P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\Omega^-\right)\right)\inf_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \\ &= E^{P_e}\left(\overline{R}_{1,e}\right)\inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\overline{R}_{1,e}\right)(\tilde{\omega}) \\ &= 0. \end{aligned}$$

$\square$

Appendix C

Calibrated parameters:

$$\pi^* = \begin{bmatrix} 0.1414 & 0.8200 & 0.0309 & 0.0077 \\ 0.2637 & 0.6820 & 0.0486 & 0.0057 \\ 0.0309 & 0.0077 & 0.1414 & 0.8200 \\ 0.0486 & 0.0057 & 0.2637 & 0.6820 \end{bmatrix}$$

s_t	Endowments	Growth Rates
1	0.6438	0.9602
2	0.6438	1.0402
3	0.3562	0.9602
4	0.3562	1.0402

C.1 Standard Model I: CE Allocations

The calibrated β is given by $\frac{1}{1.008}\sum_{\xi}\psi_{po}(\xi)\left(\sum_{\xi'}\pi_h(\xi'|\xi)g(\xi')^{-\sigma}\right)^{-1}$.

C.2 Standard Model II: CESC allocations - Homogeneous Beliefs

The invariant distribution is

State	Probability
$\left[1,\left(\underline{\alpha}_{1,cpo}\left(1,\mu^{\pi^*}\right),1-\underline{\alpha}_{1,cpo}\left(1,\mu^{\pi^*}\right)\right)\right]$	0.1370
$\left[2,\left(\underline{\alpha}_{1,cpo}\left(2,\mu^{\pi^*}\right),1-\underline{\alpha}_{1,cpo}\left(2,\mu^{\pi^*}\right)\right)\right]$	0.0098
$\left[2,\left(\underline{\alpha}_{1,cpo}\left(1,\mu^{\pi^*}\right),1-\underline{\alpha}_{1,cpo}\left(1,\mu^{\pi^*}\right)\right)\right]$	0.3532
$\left[3,\left(1-\underline{\alpha}_{2,cpo}\left(3,\mu^{\pi^*}\right),\underline{\alpha}_{2,cpo}\left(3,\mu^{\pi^*}\right)\right)\right]$	0.1370
$\left[4,\left(1-\underline{\alpha}_{2,cpo}\left(4,\mu^{\pi^*}\right),\underline{\alpha}_{2,cpo}\left(4,\mu^{\pi^*}\right)\right)\right]$	0.0098
$\left[4,\left(1-\underline{\alpha}_{2,cpo}\left(3,\mu^{\pi^*}\right),\underline{\alpha}_{2,cpo}\left(3,\mu^{\pi^*}\right)\right)\right]$	0.3532

$$\tilde{\theta}_{AJ} = \left(\frac{1-\underline{\tilde{\alpha}}_{2,cpo}\left(3,\mu^{\pi^*}\right)}{\underline{\tilde{\alpha}}_{1,cpo}\left(1,\mu^{\pi^*}\right)}\right)^{-\sigma} > 1 \text{ for } \epsilon(\xi') \neq \epsilon(\xi)$$

C.3 CESC allocations - Heterogeneous Beliefs

$$\theta_{AJ}(\xi'|\xi,\alpha) = \begin{cases} \left(\frac{1-\underline{\alpha}_{1,cpo}(1,\mu^{\pi})}{1-\alpha_1}\right)^{-\sigma} > 1 & \text{if } \xi'=1 \;\&\; \xi\in\{3,4\} \quad (\epsilon(\xi')\neq\epsilon(\xi)) \\ \left(\frac{1-\underline{\alpha}_{2,cpo}(3,\mu^{\pi})}{\alpha_1}\right)^{-\sigma} > 1 & \text{if } \xi'=3 \;\&\; \xi\in\{1,2\} \quad (\epsilon(\xi')\neq\epsilon(\xi)) \\ 1 & \text{o.w.} \end{cases}$$

and

$$\theta_{BH}(\xi'|\xi,\alpha) = \begin{cases} \left[\min\left(\frac{\pi_2(1|2)}{\alpha_1\pi^*(1|2)+(1-\alpha_1)\pi_2(1|2)},\frac{1-\underline{\alpha}_{1,cpo}(1,\mu^{\pi})}{1-\alpha_1}\right)\right]^{-\sigma} < 1 & \text{if } \xi'=1 \;\&\; \xi=2 \quad (\epsilon(\xi')=\epsilon(\xi)) \\ \left[\min\left(\frac{\pi_2(3|4)}{\alpha_1\pi^*(3|4)+(1-\alpha_1)\pi_2(3|4)},\frac{1-\underline{\alpha}_{1,cpo}(3,\mu^{\pi})}{1-\alpha_1}\right)\right]^{-\sigma} < 1 & \text{if } \xi'=3 \;\&\; \xi=4 \quad (\epsilon(\xi')=\epsilon(\xi)) \end{cases}$$

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Supplementary Material

Let $c_i(\xi,\alpha)$ and $w_i'(\xi,\alpha,\mu)(\xi^{'})$ be the maximisers in problem (6) - (10) and let $\lambda_i(\xi,\alpha,\mu)$ be the Lagrange multiplier associated to constraint (8). Let

$$\tilde{u}_i(\xi,\alpha,\mu) = u_i(c_i(\xi,\alpha)) + \beta(\xi,\mu_i)\sum_{\xi'}\pi_{\mu_i}(\xi'\,|\xi\,)\; w_i'(\xi,\alpha,\mu)(\xi').$$

Claim 1. *$\tilde{u}_i(\xi,\alpha,\mu)$ is nondecreasing in α_i for all $\alpha \in \mathbb{R}_+^I$.*

Proof. Let $\tilde{\alpha},\alpha \in \mathbb{R}_+^I$ be such that $\tilde{\alpha}_i > \alpha_i$ and $\tilde{\alpha}_j = \alpha_j$ for every $j \neq i$. To get a contradiction, suppose $\tilde{u}_i(\xi,\tilde{\alpha},\mu) < \tilde{u}_i\,(\xi,\alpha,\mu)$. Since the constrained set is independent of the welfare weights, then

$$\sum_h \tilde{\alpha}_h\left(\tilde{u}_h\left(\xi,\tilde{\alpha},\mu\right) - \tilde{u}_h\left(\xi,\alpha,\mu\right)\right) \geq 0 \text{ and } \sum_h \alpha_h\left(\tilde{u}_h\left(\xi,\alpha,\mu\right) - \tilde{u}_h\left(\xi,\tilde{\alpha},\mu\right)\right) \geq 0$$

and so, on the one hand,

$$\sum_h(\tilde{\alpha}_h - \alpha_h)\left(\tilde{u}_h\left(\xi,\tilde{\alpha},\mu\right) - \tilde{u}_h\left(\xi,\alpha,\mu\right)\right) \geq 0$$

But, on the other hand,

$$\sum_h(\tilde{\alpha}_h - \alpha_h)\left(\tilde{u}_h\left(\xi,\tilde{\alpha},\mu\right) - \tilde{u}_h\left(\xi,\alpha,\mu\right)\right) = (\tilde{\alpha}_i - \alpha_i)\left(\tilde{u}_i\left(\xi,\tilde{\alpha},\mu\right) - \tilde{u}_i\left(\xi,\alpha,\mu\right)\right) < 0$$

a contradiction. $\square$

Let $\overline{c}_i(\xi,\alpha)$ and $\overline{w}_i'(\xi,\alpha,\mu)(\xi^{'})$ be the maximisers of the relaxed problem where (8) is ignored. Let

$$\overline{u}(\xi,\alpha,\mu) = u_i(\overline{c}_i(\xi,\alpha)) + \beta(\xi,\mu_i)\sum_{\xi'}\pi_{\mu_i}(\xi'\,|\xi\,)\;\overline{w}_i'(\xi,\alpha,\mu)(\xi').$$

Claim 2. *Let $\alpha \in \mathbb{R}_+^I$. If $\alpha_i < \tilde{\alpha}_i$ and $\alpha_h = \tilde{\alpha}_h$ for all $h \neq i$, then $\overline{u}_i\left(\xi,\alpha,\mu\right) < \overline{u}_i\left(\xi,\tilde{\alpha},\mu\right)$.*

Proof. Note that $\overline{c}_i(\xi,\alpha)$ is the unique solution to

$$c_i + \sum_{h\neq i}\left(\frac{\partial u_h}{\partial c_h}\right)^{-1}\left(\frac{\alpha_i}{\alpha_h}\;\frac{\partial u_i(c_i)}{\partial c_i}\right) = y(\xi).$$

and so it is strictly increasing in α_i. Therefore, $\overline{c}_i(\xi,\tilde{\alpha}) > \overline{c}_i(\xi,\alpha)$. Note that

$$\overline{\alpha}_i'(\xi,\alpha,\mu)(\xi') = \frac{\alpha_i\int\pi(\xi'\,|\xi)\mu_i'(\xi,\mu)\,(\xi')\,(d\pi)}{\sum_h\alpha_h\int\pi(\xi'\,|\xi)\mu_h'(\xi,\mu)\,(\xi')\,(d\pi)}$$

Thus, $\overline{\alpha}_i'(\xi,\alpha,\mu)(\xi')$ is nondecreasing in α_i. Since $\overline{w}_i'(\xi,\alpha,\mu)(\xi^{'})$ satisfies (9) and (10), it follows by Lemma A.1 and Theorem 1 that $\overline{w}_i'(\xi,\alpha,\mu)(\xi^{'}) = \overline{u}_i(\xi',\overline{\alpha}'(\xi,\alpha,\mu)(\xi'),\mu'(\xi,\mu)(\xi'))$. Thus, Claim 1 implies that $\overline{w}_i'(\xi,\tilde{\alpha},\mu)(\xi^{'}) \geq \overline{w}_i'(\xi,\alpha,\mu)(\xi^{'})$ for all ξ'. We conclude that $\overline{u}_i\left(\xi,\alpha,\mu\right) < \overline{u}_i\left(\xi,\tilde{\alpha},\mu\right)$, as desired. $\square$

Proof of Proposition 2 . (i) Suppose $\alpha \in \Delta(\xi, \mu)$. Consider first the case where $\alpha_i > \underline{\alpha}_i(\xi, \mu)$ for all i. By the definition of $\tilde{u}_i(\xi, \alpha, \mu)$, we have that $\tilde{u}_i(\xi, \alpha, \mu) \geq U_i(\xi, \mu)$ and $\sum_i^I \alpha_i \tilde{u}_i(\xi, \alpha, \mu) = v^*(\xi, \alpha, \mu)$. It follows by Lemma A.1, that $(\tilde{u}_1(\xi, \alpha, \mu)...\tilde{u}_I(\xi, \alpha, \mu)) \in \mathcal{U}^{\mathrm{E}}(\xi, \mu)$. Since $\sum_i^I \alpha_i \tilde{u}_i(\xi, \alpha, \mu) = v^*(\xi, \alpha, \mu)$, it is easy to see that $(u_1(\xi, \alpha, \mu)...u_I(\xi, \alpha, \mu)) \in \overline{\mathcal{U}}^E(\xi, \mu)$. Then, $\tilde{u}_i(\xi, \alpha, \mu) > U_i(\xi, \mu_i)$ for all i by definition of $\underline{\alpha}_i(\xi, \mu)$. Thus, $\lambda_i(\xi, \alpha, \mu) = 0$. Let $\alpha \in \Delta(\xi, \mu)$ be such that $\alpha_i = \underline{\alpha}_i(\xi, \mu)$ for some i. Then there is a sequence $\{\alpha^n\}_{n=1}^{\infty}$ such that $\alpha_i^n > \underline{\alpha}_i(\xi, \mu)$ for all i and n and $\alpha^n \to \alpha$. It follows that

$$\lambda_i(\xi, \alpha, \mu) = \lambda_i(\xi, \lim_{n\to\infty} \alpha^n, \mu) = \lim_{n\to\infty} \lambda_i(\xi, \alpha^n, \mu) = 0,$$

where the second equality follows by continuity of $\lambda_i(\xi, \alpha, \mu)$ in α and the last one because weak inequalities are preserved under limits. It follows that, $\tilde{u}_i(\xi, \alpha, \mu) = \overline{u}_i(\xi, \alpha, \mu)$ and so $c_i(\xi, \alpha) = \overline{c}_i(\xi, \alpha)$, i.e. $c_i(\xi, \alpha)$ solves the relaxed problem.

(ii) Let $\alpha \in \mathbb{R}_+^I$ and $\alpha^* \equiv \left(\frac{\alpha_1}{\sum_{i=1}^I \alpha_i} \cdots \frac{\alpha_I}{\sum_{i=1}^I \alpha_i}\right)$. If $\alpha^* \in \Delta(\xi, \mu)$, then $c_i(\xi, \alpha) = c_i(\xi, \alpha^*)$ because $\tilde{u}_i(\xi, \alpha, \mu)$ is homogeneous of degree zero in α. If $\alpha^* \notin \Delta(\xi, \mu)$, there is i such that $\alpha_i^* < \underline{\alpha}_i(\xi, \mu)\left(\alpha_{-i}^*\right)$.

- First, we show that $\lambda_i(\xi, \alpha, \mu) > 0$. To get a contradiction, suppose $\lambda_i(\xi, \alpha, \mu) = 0$. It follows that

$$\begin{aligned}
\tilde{u}_i\left(\xi, (\alpha_i, \alpha_{-i}), \mu\right) \quad & = \tilde{u}_i\left(\xi, \left(\alpha_i^*, \alpha_{-i}^*\right), \mu\right) \\
& = \overline{u}_i\left(\xi, \left(\alpha_i^*, \alpha_{-i}^*\right), \mu\right) \\
& = \overline{u}_i\left(\xi, \left(\frac{\alpha_i^*}{\underline{\alpha}_i(\xi, \mu)\left(\alpha_{-i}^*\right)}, \frac{\alpha_{-i}^*}{\underline{\alpha}_i(\xi, \mu)\left(\alpha_{-i}^*\right)}\right), \mu\right) \\
& < \overline{u}_i\left(\xi, \left(1, \frac{\alpha_{-i}^*}{\underline{\alpha}_i(\xi, \mu)\left(\alpha_{-i}^*\right)}\right), \mu\right) \\
& = \overline{u}_i\left(\xi, \left(\underline{\alpha}_i(\xi, \mu)\left(\alpha_{-i}^*\right), \alpha_{-i}^*\right), \mu\right) \\
& = U_i(\xi, \mu),
\end{aligned}$$

  where the first equality follows because $\tilde{u}_i$ is homogeneous of degree zero in α, the second one is due to the assumption that $\lambda_i(\xi, \alpha, \mu) = 0$ and the homogeneity of degree zero of $\lambda_i(\xi, \alpha, \mu)$ in α, the third and fifth follows by homogeneity of degree zero of $\overline{u}_i(\cdot)$ in α, the inequality follows by Claim 2 and the last equality follows by definition of the minimum enforceable weights. But then, $\tilde{u}_i\left(\xi, (\alpha_i, \alpha_{-i}), \mu\right) < U_i(\xi, \mu)$ which contradicts constraint (8).

- Second, note that problem (6) - (10) is equivalent to maximising

$$\sum_{i=1}^I (\alpha_i + \lambda_i) \left\{ u_i(c_i) + \beta(\xi, \mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\; w_i'(\xi') \right\},$$

  subject to constraints (7), (9) and (10).

- Finally, the latter is equivalent to the relaxed problem with welfare weights $\tilde{\alpha}$ given by

$$\tilde{\alpha}_i = \frac{\alpha_i + \lambda_i(\xi, \alpha, \mu)}{\sum_{h=1}^I \left(\alpha_h + \lambda_h(\xi, \alpha, \mu)\right)},$$

Thus, $\overline{u}_i\left(\xi, \tilde{\alpha}, \mu\right) = \tilde{u}_i\left(\xi, \alpha, \mu\right) \geq U_i(\xi, \mu) = \overline{u}_i\left(\xi, \underline{\alpha}_i, \mu\right)$. It follows by Claim 2 that $\tilde{\alpha}_i \geq \underline{\alpha}_i$. Therefore, $\tilde{\alpha} \in \Delta(\xi, \mu)$ and $c_i(\xi, \alpha) = \overline{c}_i(\xi, \tilde{\alpha}) = c_i(\xi, \tilde{\alpha})$ as desired. $\square$

Now we prove Theorem 11. We begin with some results on Markov Processes.

Lemma 7.1. *Let $\{z_t\}_{t=0}^{\infty}$ be a two-state time homogeneous Markov process with transition function F on $(Z, \mathcal{Z})$ and invariant distribution $\psi : \mathcal{Z} \to [0,1]$, P^F be the probability measure on $(Z^\infty, \mathcal{Z}^\infty)$ uniquely induced by F and ψ and let $R : Z \times Z \to \Re$. Suppose there exists $z_+ \in Z$ such that*
(a) $E^{P^F}(R(z_1, z_2)) = 0$.
(b) $R(z, z_+) > 0$ for all z.
(c) $E^{P^F}(R(z_0, z_1)\ R(z_1, z_2)) > 0$.
Then $E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) < 0 < E^{P^F}(R(z_2, z_3)|\, z_1 = z_-)$ iff $F(z_+|\, z_+) < \psi(z_+)$.

Proof. Hypothesis (a) and the Markov property implies that $E^{P^F}(R(z_k, z_{k+1})) = 0$ for any k. Thus,

$$\psi(z-)\ E^{P^F}(R(z_{k'}, z_{k'+1})|\, z_k = z_-) = -\psi(z_+)\ E^{P^F}(R(z_{k'}, z_{k'+1})|\, z_k = z_+) \tag{34}$$

where $z_- \neq z_+$. Note also that

$$\begin{aligned} & E^{P^F}(R(z_0, z_1)\ R(z_1, z_2)) = E^{P^F}\Big(R(z_0, z_1)\ E^{P^F}(R(z_1, z_2)|\, z_1)\Big) \\ = & \left[P^F(z_+, z_+) R(z_+, z_+) + P^F(z_-, z_+) R(z_-, z_+)\right]\ E^{P^F}(R(z_1, z_2)|\, z_1 = z_+) \\ & + \left[P^F(z_+, z_-) R(z_+, z_-) + P^F(z_-, z_-) R_e(z_-, z_-)\right]\ E^{P^F}(R(z_1, z_2)|\, z_1 = z_-). \end{aligned} \tag{35}$$

By hypothesis (a) and (b), $R(z, z_-) < 0$ for all z. Therefore,

$$\begin{aligned} P^F(z_+, z_-) R(z_+, z_-) + P^F(z_-, z_-) R(z_-, z_-) &< 0, \\ P^F(z_+, z_+) R(z_+, z_+) + P^F(z_-, z_+) R(z_-, z_+) &> 0. \end{aligned}$$

It follows from (34) evaluated at $k = 1$ and $k' = 1$, hypothesis (c) and (35) that

$$E^{P^F}(R(z_1, z_2)|\, z_1 = z_-) < 0 < E^{P^F}(R(z_1, z_2)|\, z_1 = z_+)$$

and the Markov Property implies

$$E^{P^F}(R(z_2, z_3)|\, z_2 = z_-) < 0 < E^{P^F}(R(z_2, z_3)|\, z_2 = z_+). \tag{36}$$

Condition (34), evaluated at $k = 1$ and $k' = 2$, implies that

$$E^{P^F}(R(z_2, z_3)|\, z_1 = z_-) < 0 < E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) \Leftrightarrow E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) > 0.$$

In addition,

$$\begin{aligned} E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) &= E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) - E^{P^F}(R(z_2, z_3)) \\ &= (F(z_2 = z_+|\, z_1 = z_+) - \psi(z_+))\, E^{P^F}(R(z_2, z_3)|\, z_2 = z_+) + \\ &\quad (F(z_2 = z_-|\, z_1 = z_+) - \psi(z_-))\, E^{P^F}(R(z_2, z_3)|\, z_2 = z_-) \\ &= (F(z_2 = z_+|\, z_1 = z_+) - \psi(z_+)) \times \\ &\quad \Big(E^{P^F}(R(z_2, z_3)|\, z_2 = z_+) - E^{P^F}(R(z_2, z_3)|\, z_2 = z_-)\Big). \end{aligned}$$

where the first line follows by the definition of unconditional expectation and (a). (36) implies that

$$E^{P^F}(R(z_2, z_3)|\, z_1 = z_+) < 0 \Leftrightarrow F(z_2 = z_+|\, z_1 = z_+) - \psi(z_+) < 0.$$

□

Proof of Theorem 11(a). Consider any CE of an arbitrary baseline growth economy. Since the allocation is PO, it follows by Theorem 8 that (15) holds and the marginal distribution of ψ_{po} over welfare weights is a point mass on α_∞. By standard arguments, there exists $\overline{R}_{po} : \{l,h\} \times \{l,h\} \to \Re$ such that for any $\tau \in \{1,2\}$ and $\omega \in \Omega$

$$\overline{R}_{\tau,po}(\omega) = \begin{cases} \overline{R}_{po}(l,l) & \text{if } \xi_{\tau-1}(\omega) \in \{1,3\} \text{ and } \xi_\tau(\omega) \in \{1,3\} \\ \overline{R}_{po}(l,h) & \text{if } \xi_{\tau-1}(\omega) \in \{1,3\} \text{ and } \xi_\tau(\omega) \in \{2,4\} \\ \overline{R}_{po}(h,l) & \text{if } \xi_{\tau-1}(\omega) \in \{2,4\} \text{ and } \xi_\tau(\omega) \in \{1,3\} \\ \overline{R}_{po}(h,h) & \text{if } \xi_{\tau-1}(\omega) \in \{2,4\} \text{ and } \xi_\tau(\omega) \in \{2,4\} \end{cases}$$

and

$$\overline{R}_{po}(\xi, l) < 0 < \overline{R}_{po}(\xi, h) \text{ for all } \xi \in \{l,h\}. \tag{37}$$

Let $Z = \{l,h\}$, $\mathcal{Z}$ be its finest partition, $\tilde{\pi}^*$ be the transition function on $(Z,\mathcal{Z})$ defined as the restriction of π^* to $(Z,\mathcal{Z})$ and let $\tilde{\psi}_{po}$ be the restriction of the invariant measure ψ_{po} to $(Z,\mathcal{Z})$. Let Z^∞ be the set of infinite sequences with elements in Z and $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset ... \subset \mathcal{Z}_t \subset ...\mathcal{Z}^\infty$ be the standard filtration. $P^{\tilde{\pi}^*}$ is the probability measure over $(Z^\infty, \mathcal{Z}^\infty)$ uniquely induced by $\tilde{\pi}^*$ and $\tilde{\psi}_{po}$. Let $z_t : Z^\infty \to Z$ be $\mathcal{Z}_t-$measurable. The collection $\{z_t\}_{t=0}^\infty$ on the probability space $\left(Z^\infty, \mathcal{Z}^\infty, P^{\tilde{\pi}^*}\right)$ is a two state time-homogeneous Markov process with transition function $\tilde{\pi}^*$ on $(Z,\mathcal{Z})$ and invariant distribution $\tilde{\psi}_{po} : \mathcal{Z} \times \mathcal{Z} \to [0,1]$ satisfying

$$E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_1, z_2)\right) = 0. \tag{38}$$

First note that (38) and (37) are conditions (a) and (b), respectively, in Lemma 7.1. Second, since the asset displays short-term momentum,

$$0 < E^{P_{po}}\left(\overline{R}_{1,po}\ \overline{R}_{2,po}\right) = E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_0, z_1)\ \overline{R}_{po}(z_1, z_2)\right)$$

and so condition (c) in Lemma 7.1 also holds. By Lemma 7.1, we conclude that

$$E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_2, z_3)\middle|\, z_1 = h\right) < 0 < E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_2, z_3)\middle|\, z_1 = l\right) \Leftrightarrow \tilde{\pi}^*(h|\,h) < \tilde{\psi}_{po}(h). \tag{39}$$

Let ω^+ and ω^- be such that $\overline{R}_{1,po}(\omega^+) > 0$ and $\overline{R}_{1,po}(\omega^-) < 0$. Then,

$$\begin{aligned} E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)(\omega^+) &= E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_2, z_3)\middle|\, z_1 = h\right), \\ E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)(\omega^-) &= E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{po}(z_2, z_3)\middle|\, z_1 = l\right). \end{aligned}$$

It follows from (39), $\tilde{\pi}^*(h|\,h) = \pi^*(2|\,2) + \pi^*(4|\,2)$ and $\tilde{\psi}_{po}(h) = \psi_{po}(2) + \psi_{po}(4)$ that

$$E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)(\omega^+) < 0 < E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)(\omega^-) \Leftrightarrow \pi^*(2|\,2) + \pi^*(4|\,2) < \psi_{po}(2) + \psi_{po}(4)$$

that is, $E^{P_{po}}\left(\overline{R}_{3,po}\middle|\,\overline{R}_{1,po}\right)$ reverts to the mean if and only if $\pi^*(2|\,2) + \pi^*(4|\,2) < \psi_{po}(2) + \psi_{po}(4)$. By Proposition 9, the asset displays long-term reversal if $\pi^*(2|\,2) + \pi^*(4|\,2) < \psi_{po}(2) + \psi_{po}(4)$. To show the converse, suppose that $\pi^*(2|\,2) + \pi^*(4|\,2) \geq \psi_{po}(2) + \psi_{po}(4)$. Then by the argument above, $E^{P_e}\left(R_{3,po}\middle|\,\overline{R}_{1,e}\right)$ trends and it follows by Proposition 9 that the 2nd-order autocorrelation is positive and so long-run reversal fails. $\square$

Proof of Theorem 11(b). Consider any CESC of an arbitrary baseline growth economy. The price of an asset at state (ξ, α) must satisfy the Bellman equation:

$$p(\xi, \alpha) = \sum_{\xi'} Q(\xi, \alpha)(\xi')\left(p(\xi', \alpha'(\xi, \alpha)(\xi')) + d(\xi')\right) \qquad \psi_{cpo} - a.s.$$

It is easy to see that the invariant distribution places positive mass only on points (ξ, α) such that $\alpha \in \underline{\Delta} \cap \Delta\left(\xi, \mu^{\pi^*}\right)$ where $\underline{\Delta} = \{(\alpha_1, \alpha_2) \in \Delta : \exists \xi \in S \text{ such that } \alpha_1 = \underline{\alpha}_1(\xi) \text{ or } \alpha_2 = \underline{\alpha}_2(\xi)\}$. The hypothesis $\underline{\alpha}_1(1) = \underline{\alpha}_1(2)$ and symmetry implies that $\underline{\alpha}_2(3) = \underline{\alpha}_2(4)$. If p_ξ, $q_{\xi\xi'}$ and d_ξ denotes $p(\xi, \underline{\alpha}(\xi))$, $Q(\xi, \underline{\alpha}(\xi))(\xi')$ and $d(\xi)$, respectively, then the Bellman equation becomes

$$p_\xi = \sum_{\xi'} q_{\xi\xi'}(p_{\xi'} + d_{\xi'}) \quad \text{for all } \xi$$

which can be written as $(I - Q)P = QD$ where Q is the 4×4 matrix with entries $q_{\xi\xi'}$, P is the 4×1 vector with entries p_ξ and D is the 4×1 vector with entries p_ξ. Note that

$$c_1(1, \underline{\alpha}(1)) = c_2(3, \underline{\alpha}(3)) \quad \text{and} \quad c_1(2, \underline{\alpha}(2)) = c_2(4, \underline{\alpha}(4))$$

and so

$$\begin{aligned} q_{\xi 1} &= \beta(\xi, \mu)\, \pi(1\,|\xi)\, \tfrac{\partial u(c_1(1, \underline{\alpha}(1))/\partial c_1}{\partial u(c_1(\xi, \underline{\alpha}(\xi)))/\partial c_1} = \beta(\xi, \mu)\, \pi(3\,|\xi)\, \tfrac{\partial u(c_2(3, \underline{\alpha}(3))/\partial c_1}{\partial u(c_2(\xi, \underline{\alpha}(\xi)))/\partial c_1} = q_{\xi 3}, \\ q_{\xi 2} &= \beta(\xi, \mu)\, \pi(2\,|\xi)\, \tfrac{\partial u(c_1(2, \underline{\alpha}(2))/\partial c_1}{\partial u(c_1(\xi, \underline{\alpha}(\xi)))/\partial c_1} = \beta(\xi, \mu)\, \pi(4\,|\xi)\, \tfrac{\partial u(c_2(4, \underline{\alpha}(4))/\partial c_1}{\partial u(c_2(\xi, \underline{\alpha}(\xi)))/\partial c_1} = q_{\xi 4}. \end{aligned}$$

It follows that Q has rank 2. Therefore, $p_1 = p_3$ and $p_2 = p_4$.

Let $\tilde{\pi}^*$ and $(Z^\infty, \mathcal{Z}^\infty)$ be the transition matrix and the measurable space, respectively, introduced in the proof of Theorem 11(a). $P^{\tilde{\pi}^*}$ is the probability measure over $(Z^\infty, \mathcal{Z}^\infty)$ uniquely induced by $\tilde{\pi}^*$ and $\tilde{\psi}_{cpo}$. Let $z_t : Z^\infty \to Z$ be $\mathcal{Z}_t-$measurable. The collection $\{z_t\}_{t=0}^\infty$ on the probability space $\left(Z^\infty, \mathcal{Z}^\infty, P^{\tilde{\pi}^*}\right)$ is a two state time-homogeneous Markov process with transition function $\tilde{\pi}^*$ on $(Z, \mathcal{Z})$ and invariant distribution $\tilde{\psi}_{cpo} : \mathcal{Z} \times \mathcal{Z} \to [0, 1]$.

Let $p(l) \equiv p_1$, $p(h) \equiv p_2$, $R_{cpo}(z, z') \equiv \frac{p_{z'} + d_{z'}}{p_z}$ for all $z \in \{l, h\}$ and $\overline{R}_{cpo} : \{l, h\} \times \{l, h\} \to \Re$ be such that

$$\overline{R}_{\tau,cpo}(\omega) = \begin{cases} \overline{R}_{cpo}(l, l) & \text{if } \xi_{\tau-1}(\omega) \in \{1, 3\} \text{ and } \xi_\tau(\omega) \in \{1, 3\} \\ \overline{R}_{cpo}(l, h) & \text{if } \xi_{\tau-1}(\omega) \in \{1, 3\} \text{ and } \xi_\tau(\omega) \in \{2, 4\} \\ \overline{R}_{cpo}(h, l) & \text{if } \xi_{\tau-1}(\omega) \in \{2, 4\} \text{ and } \xi_\tau(\omega) \in \{1, 3\} \\ \overline{R}_{cpo}(h, h) & \text{if } \xi_{\tau-1}(\omega) \in \{2, 4\} \text{ and } \xi_\tau(\omega) \in \{2, 4\} \end{cases} \tag{40}$$

Moreover,

$$\overline{R}_{cpo}(z, l) < 0 < \overline{R}_{cpo}(z, h) \text{ for all } z \in \{l, h\} \tag{41}$$

and

$$E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{cpo}(z_1, z_2)\right) = 0. \tag{42}$$

It follows from (40) that for any $k \in \{2, 3\}$

$$E^{P_{cpo}}\left(\overline{R}_{1,cpo}\ \overline{R}_{k,cpo}\right) = E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{cpo}(z_0, z_1)\ \overline{R}_{cpo}(z_1, z_k)\right).$$

Note that (42) and (41) are conditions (a) and (b) in Lemma 7.1. Since the asset displays short-term momentum,

$$E^{P^{\tilde{\pi}^*}}\left(\overline{R}_{cpo}(z_0, z_1)\ \overline{R}_{cpo}(z_1, z_2)\right) = E^{P_{cpo}}\left(\overline{R}_{1,po}\ \overline{R}_{2,po}\right) > 0,$$

and so (c) in Lemma 7.1 also holds. The rest of the proof is identical to that in Theorem 11(a). $\square$