An Introduction to Market Microstructure Invariance

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HSE, Moscow
November 8, 2014

Overview

Our goal is to explain how **order size**, **order frequency**, **market efficiency and trading costs** vary across time and stocks.

- We propose **market microstructure invariance** that generates predictions concerning variations of these variables.
- We develop a meta-model suggesting that invariance is ultimately related to granularity of information flow.
- Invariance relationships are tested using a data set of portfolio transitions and find a strong support in the data.
- Invariance implies simple formulas for order size, order frequency, market efficiency, market impact, and bid-ask spread as functions of observable volume and volatility.

Preview of Results: Bet Sizes

Our estimates imply that bets $|\tilde{X}|/V$ are approximately distributed as **a log-normal** with the log-variance of 2.53 and the number of bets per day γ is defined as $(W = V \cdot P \cdot \sigma)$,

$$\ln \gamma = \ln 85 + \frac{2}{3} \ln \Big[\frac{W}{(0.02)(40)(10^6)} \Big].$$

$$\ln \Big[\frac{|\tilde{X}|}{V} \Big] \approx -5.71 - \frac{2}{3} \cdot \ln \Big[\frac{W}{(0.02)(40)(10^6)} \Big] + \sqrt{2.53} \cdot N(0, 1)$$

For a benchmark stock, there are 85 bets with the median size of 0.33% of daily volume. Buys and sells are symmetric.

Preview of Results: Transaction Costs

Our estimates imply two simple formulas for expected trading costs for any order of X **shares** and for any security. The linear and square-root specifications are:

$$C(X) = \left(\frac{W}{(0.02)(40)(10^6)}\right)^{-1/3} \frac{\sigma}{0.02}\left(\frac{2.50}{10^4}\cdot\frac{X}{0.01V}\left[\frac{W}{(0.02)(40)(10^6)}\right]^{2/3}+\frac{8.21}{10^4}\right).$$

$$C(X) = \left(\frac{W}{(0.02)(40)(10^6)}\right)^{-1/3} \frac{\sigma}{0.02}\left(\frac{12.08}{10^4}\cdot\sqrt{\frac{X}{0.01V}\left[\frac{W}{(0.02)(40)(10^6)}\right]^{2/3}}+\frac{2.08}{10^4}\right).$$

A Structural Model

We outline a dynamic infinite-horizon model of trading, from which various invariance relationships are derived results.

- **Informed traders** face given costs of acquiring information of given precision, then place informed bets which incorporate a given fraction of the information into prices.
- **Noise traders** place bets which turn over a constant fraction of the stocks float, mimicking the size distribution of bets placed by informed trades.
- **Market makers** offer a residual demand curve of constant slope, lose money from being "run over" by informed bets, but make up the losses from trading costs imposed on informed and noise traders.

Fundamental Value

- The unobserved "fundamental value" of the asset follows an exponential martingale:

$$F(t) := \exp[\sigma_F \cdot B(t) - \tfrac{1}{2} \cdot \sigma_F^2 \cdot t],$$

where $B(t)$ follows standardized Brownian motion with $\mathrm{var}\{B(t + \Delta t) - B(t)\} = \Delta t$. $F(t)$ follows a martingale.

Market Prices

- The price changes as informed traders and noise traders arrive in the market and anonymously place bets.
- Risk neutral market makers set the market price $P(t)$ as the conditional expectation of the fundamental value $F(t)$ given a history of the "bet flow".
- $\bar{B}(t)$ is the market's conditional expectation of $B(t)$ based on observing the history of prices; the error $B(t) - \bar{B}(t)$ has a normal distribution with variance denoted $\Sigma(t)/\sigma_F^2$.
- The price is the best estimate of fundamental value; the price has a martingale property:

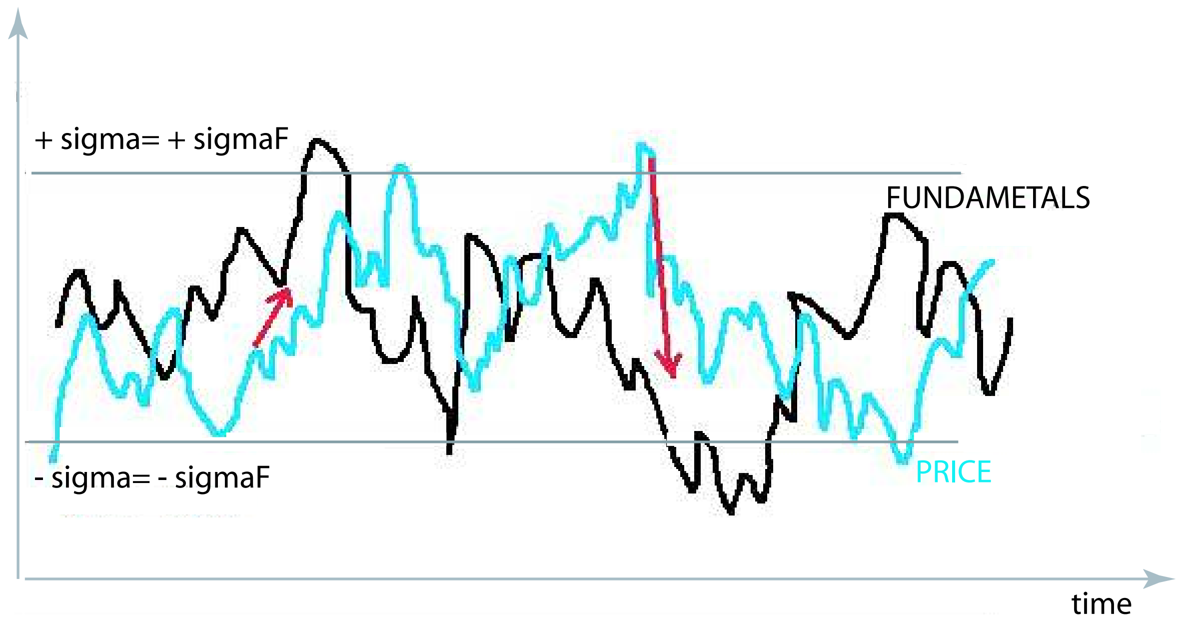
$$P(t) = \exp[\sigma_F \cdot \bar{B}(t) + \tfrac{1}{2} \cdot \Sigma(t) - \tfrac{1}{2} \cdot \sigma_F^2 \cdot t].$$

Pricing Accuracy

- Pricing accuracy is defined as $\Sigma(t) = var\{\log[F(t)/P(t)]\}$; market is more efficient when $\Sigma^{1/2}$ is smaller.

- $\Sigma^{-1/2}$ is Fischer Black's measure of market efficiency: He conjectures *"almost all markets are efficient" in the sense that "price is within a factor 2 of value" at least* 90% *of the time.*

Pricing Accuracy - Intuition



- Pricing accuracy is defined as $\Sigma(t) = var\{\log[F(t)/P(t)]\}$; the market is more efficient when $\Sigma^{1/2}$ is smaller.
- Fama says a market is "efficient" if all information is appropriately reflected in price (prices follow a martingale), even if very little information is available and prices are not very accurate, i.e., $\Sigma^{1/2}$ is large.

Pricing Accuracy

- $\Sigma^{-1/2}$ is Fischer Black's measure of market efficiency: He conjectures *"almost all markets are efficient" in the sense that "price is within a factor 2 of value" at least* 90% *of the time.* In mathematical terms, $\Sigma^{1/2} = \ln(2)/1.64 = 0.42$.
- In time units, Σ/σ^2 is the number of years by which the informational content of prices lags behind fundamental value, e.g., if $\sigma = 0.35$ and $\Sigma^{1/2} = \ln(2)/1.64$, then prices are about $(\ln(2)/1.64)^2/0.35^2 \approx 1.50$ years "behind" fundamental value.

Informed Traders

- Informed traders arrive randomly in the market at rate $\gamma_I(t)$.
- Each informed trader observes one private signal $\tilde{i}(t)$ and places one and only one bet, which is executed by trading with market makers.

$$\tilde{i}(t) := \tau^{1/2} \cdot \Sigma(t)^{-1/2} \cdot \sigma_F \cdot [B(t) - \bar{B}(t)] + \tilde{Z}_I(t),$$

where τ measures the precision of the signal and $\tilde{Z}_I(t) \sim N(0,1)$. $\mathrm{var}\{\tilde{i}(t)\} = 1 + \tau \approx 1$.

Informed Traders

- An informed trader updates his estimate of $B(t)$ from $\bar{B}(t)$ to $\bar{B}(t) + \Delta \bar{B}_I(t)$. Assuming τ is small,

$$\Delta \bar{B}_I(t) \approx \tau^{1/2} \cdot \Sigma(t)^{1/2} / \sigma_F \cdot \tilde{i}(t).$$

- If the signal value were to be fully incorporated into prices, then the dollar price change would be equal to

$$E\{F(t) - P(t) \,|\, \Delta \bar{B}_I(t)\} \approx P(t) \cdot \sigma_F \cdot \Delta \bar{B}_I(t).$$

- Only a fraction θ of the "fully revealing" impact is incorporated into prices ($\lambda(t)$ is price impact), i.e.,

$$\tilde{Q}(t) = \theta \cdot \lambda(t)^{-1} \cdot P(t) \cdot \sigma_F \cdot \Delta \bar{B}_I(t).$$

Profits of Informed Traders

- An informed trader's expected "paper trading" profits are

$$\bar{\pi}_I(t) := E\{[F(t) - P(t)] \cdot \tilde{Q}(t)\} = \frac{\theta \cdot P(t)^2 \cdot \sigma_F^2 \cdot E\{\Delta \bar{B}_I(t)^2\}}{\lambda(t)}.$$

- His expected profits net of costs conditional on $\Delta \bar{B}_I(t)$ are

$$E\{[F(t) - P(t)] \cdot \tilde{Q}(t) - \lambda(t) \tilde{Q}(t)^2\} = \frac{\theta(1-\theta) P(t)^2 \sigma_F^2 \cdot \Delta \bar{B}_I(t)^2}{\lambda(t)}.$$

- $\theta = 1/2$ maximizes the expected profits of the risk-neutral informed trader. We assume $0 < \theta < 1$ to accommodate possibility of informed traders being risk averse and information can be leaked.

Noise Traders

- Noise traders arrive at an endogenously determined rate $\gamma_U(t)$.
- Each noise trader places one bet which mimics the size distribution of an informed bet, even though it contains no information, i.e., $\tilde{i}(t) = \tilde{Z}_U(t) \sim N(0, 1+\tau) \approx N(0,1)$.
- Noise traders turn over a constant fraction η of shares outstanding N. The expected share volume $V(t)$ and total number of bets per day $\gamma(t) := \gamma_I(t) + \gamma_U(t)$ satisfy

$$\gamma_U(t) \cdot E\{|\tilde{Q}(t)|\} = \eta \cdot N, \qquad \gamma(t) \cdot E\{|\tilde{Q}(t)|\} = V(t).$$

Transaction Costs

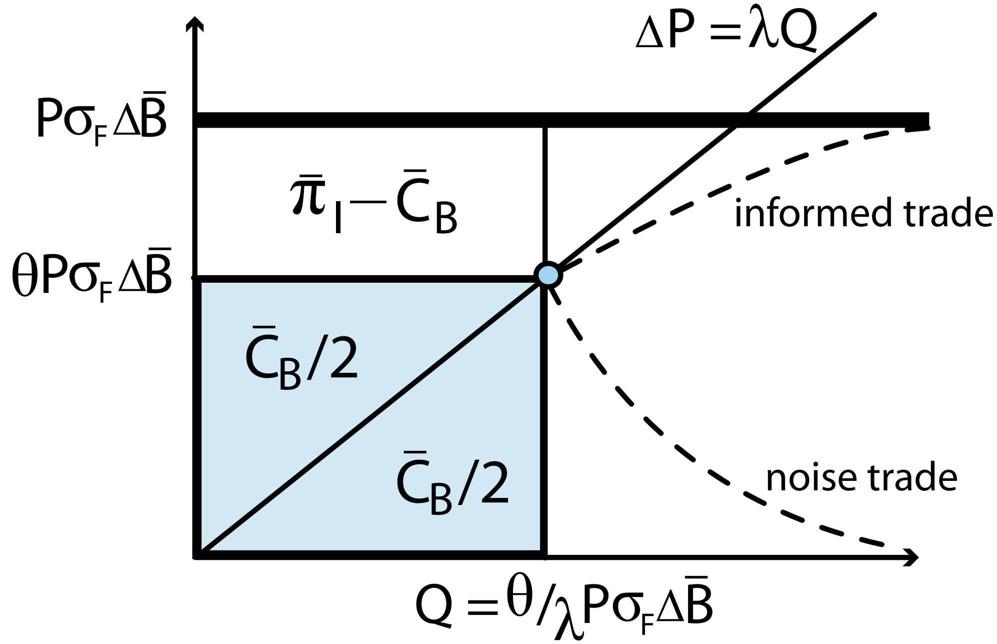
- Both informed traders and noise traders incur transactions costs. The unconditional expected costs are

$$\bar{C}_B(t) := \lambda(t) \cdot E\{\tilde{Q}(t)^2\} = \frac{\theta^2 \cdot P(t)^2 \cdot \sigma_F^2 \cdot E\{\Delta \bar{B}(t)^2\}}{\lambda(t)}.$$

- Illiquidity $1/L(t)$ is defined as the expected cost of executing a bet in basis points:

$$1/L(t) := \bar{C}_B(t)/E\{|P(t) \cdot \tilde{Q}(t)|\}.$$

Break-Even Conditions - Intuition



There is price continuation after an informed trade and mean reversion after a noise trade. The losses on trading with informed traders are equal to total gains on trading with noise traders, $\gamma_I \cdot (\bar{\pi}_I - \bar{C}_B) = \gamma_U \cdot \bar{C}_B$.

Break-Even Condition For Market Maker

- The equilibrium level of costs must allow market makers to break even.
- The expected dollar price impact costs that market makers expect to collect from all traders must be equal to the expected dollar paper trading profits of informed traders:

$$(\gamma_I(t) + \gamma_U(t)) \cdot \bar{C}_B(t) = \gamma_I \cdot \bar{\pi}_I(t).$$

Break-Even Condition for Informed Traders

- The break-even condition for informed traders yields the rate at which informed traders place bets $\gamma_I(t)$.

- The expected paper trading profits from trading on a signal $\bar{\pi}_I(t)$ must equal the sum of expected transaction costs $\bar{C}_B(t)$ and the exogenously constant cost of acquiring private information denoted c_I:

$$\bar{\pi}_I(t) = \bar{C}_B(t) + c_I.$$

Market Makers and Market Efficiency

- Zero-profit, risk neutral, competitive market makers set prices such that the price impact of anonymous trades reveals on average the information in the order flow. The average impact of a bet must satisfy

$$\lambda(t) \cdot \tilde{Q}(t) = \frac{\gamma_I(t)}{\gamma_I(t) + \gamma_U(t)} \cdot \lambda(t) \cdot \tilde{Q}(t) \cdot \frac{1}{\theta} + \frac{\gamma_U(t)}{\gamma_I(t) + \gamma_U(t)} \cdot 0.$$

- The ratio of informed traders to noise traders then turns out to be equal to the exogenous constant θ. The turnover rate is constant.

$$\frac{\gamma_I(t)}{\gamma_I(t) + \gamma_U(t)} = \theta, \quad V = \eta \cdot N/(1-\theta).$$

Diffusion Approximation

- As a result of each bet, market makers update their estimate of $\bar{B}(t)$ by $\Delta\bar{B}(t)$.
- A trade is informed with probability θ and, if informed, incorporates a fraction θ of its information content into prices, leading to an adjustment in $\bar{B}(t)$ of

$$\Delta\bar{B}(t) = \theta\tau^{1/2}\Sigma(t)^{1/2}\sigma_F^{-1}\cdot\left(\tau^{1/2}\Sigma(t)^{-1/2}\sigma_F[B(t)-\bar{B}(t)]+\tilde{Z}_I(t)\right)$$

- A trade is uninformed with probability $1-\theta$ and, if uninformed, adds noise to $\bar{B}(t)$ of

$$\Delta\bar{B}(t) = \theta\tau^{1/2}\Sigma(t)^{1/2}\sigma_F^{-1}\cdot\tilde{Z}_U(t).$$

Diffusion Approximation

- When the arrival rate of bets $\gamma(t)$ per day is sufficiently large, the diffusion approximation for the dynamics of the estimate $\bar{B}(t)$ can be written as

$$d\bar{B}(t) = \gamma(t) \cdot \theta^2 \cdot \tau \cdot [B(t) - \bar{B}(t)] \cdot dt + \gamma(t)^{1/2} \cdot \theta \cdot \tau^{1/2} \cdot \Sigma(t)^{1/2} \cdot \sigma_F^{-1} \cdot dZ(t).$$

The first term corresponds to the information contained in informed signals which arrive at rate $\theta \cdot \gamma(t)$. The second term corresponds to the noise contained in all bets arriving at rate $\gamma(t)$.

Equilibrium Price Process

- Define

$$\sigma(t) := \theta \cdot \tau^{1/2} \cdot \Sigma(t)^{1/2} \cdot \gamma(t)^{1/2}.$$

- By applying Ito's lemma,

$$\frac{dP(t)}{P(t)} = \tfrac{1}{2} \cdot [\Sigma'(t) - \sigma_F^2 + \sigma(t)^2] \cdot dt + \sigma_F \cdot d\bar{B}(t).$$

- Market efficiency implies that $P(t)$ must follow a martingale:

$$\frac{d\Sigma(t)}{dt} = \sigma_F^2 - \sigma(t)^2.$$

Price Process

- Since in the equilibrium,

$$\frac{dP(t)}{P(t)} = \sigma(t) \cdot d\bar{Z}(t).$$

The process $\bar{Z}(t)$ is a standardized Brownian motion under the market's filtration and $\sigma(t)$ is the measure of returns volatility.

Resiliency

- The difference $B(t) - \bar{B}(t)$ follows the mean-reverting process,

$$d[B(t) - \bar{B}(t)] = -\frac{\sigma(t)^2}{\Sigma(t)} \cdot [B(t) - \bar{B}(t)] \cdot dt + dB(t) - \frac{\sigma(t)}{\sigma_F} \cdot dZ(t).$$

- Market resiliency $\rho(t)$ be the mean reversion rate at which pricing errors disappear

$$\rho(t) = \frac{\sigma(t)^2}{\Sigma(t)}.$$

Holding returns volatility constant, resiliency is greater in markets with higher pricing accuracy.

Invariance Theorem - 1

Assume the cost c_I of generating a signal is an invariant constant and let $m := E\{|\tilde{i}(t)|\}$ define an additional invariant constant.

Then, the invariance conjectures hold: The dollar risk transferred by a bet per unit of business time is a random variable with an invariant distribution $\tilde{I}$, and the expected cost of executing a bet $\bar{C}_B$ is constant:

$$\tilde{I}(t) := P(t) \cdot \tilde{Q}(t) \cdot \frac{\sigma(t)}{\gamma(t)^{1/2}} = \bar{C}_B \cdot \tilde{i}(t).$$

$$\bar{C}_B = c_I \cdot \theta/(1-\theta).$$

Invariance Theorem -2

The number of bets per day $\gamma(t)$*, their size* $\tilde{Q}(t)$*, liquidity* $L(t)$*, pricing accuracy* $\Sigma(t)^{-1/2}$*, and market resiliency* $\rho(t)$ *are related to price* $P(t)$*, share volume* $V(t)$*, volatility* $\sigma(t)$*, and trading activity* $W(t) = P(t) \cdot V(t) \cdot \sigma(t)$ *by the following invariance relationships:*

$$\gamma(t) = \left(\frac{\lambda(t) \cdot V(t)}{\sigma(t)P(t)m}\right)^2 = \left(\frac{E\{|\tilde{Q}(t)|\}}{V(t)}\right)^{-1} = \frac{(\sigma(t)L(t))^2}{m^2} = \frac{\sigma(t)^2}{\theta^2\tau\Sigma(t)} = \frac{\rho(t)}{\theta^2\tau} = \left(\frac{W(t)}{m\bar{C}_B}\right)^{2/3}.$$

Arrival Rate — Impact — Bet Size — Liquidity — "Efficiency" — Activity

τ *is the precision of a signal,* θ *is the fraction of information* $\tilde{i}(t)$ *incorporated by an informed trade. The price follows a martingale with stochastic returns volatility* $\sigma(t) := \theta \cdot \tau^{1/2} \cdot \Sigma(t)^{1/2} \cdot \gamma(t)^{1/2}$*.*

Proof

The proof is based on the solution of the system of four equations:

- Volume condition: $\gamma(t) \cdot E\{|\tilde{Q}(t)|\} = V(t)$
- Market resiliency $\bar{c}_B = \lambda(t) \cdot E\{\tilde{Q}^2(t)\}$,
- Volatility condition: $\gamma(t) \cdot \lambda(t)^2 \cdot E\{\tilde{Q}(t)^2\} = P(t)^2 \cdot \sigma(t)^2$,
- Moments ratio: $m = \frac{E\{|\tilde{Q}(t)|\}}{[E\{\tilde{Q}(t)^2\}]^{1/2}}$.

One can think of $\gamma(t)$, $\lambda(t)$, $E\{\tilde{Q}(t)^2\}$, and $E\{|\tilde{Q}(t)|\}$ as unknown variables to be solved for in terms of known variables $V(t)$, $\bar{c}_B$, $P(t)$, and $\sigma(t)$.

Discussion

- Trading activity $W(t)$ and its components—prices $P(t)$, share volume V, and returns volatility $\sigma(t)$—are a "macroscopic" quantities, which are easy to estimate.
- The bet arrival rate $\gamma(t)$, bet size $\tilde{Q}(t)$, the average cost of a bet $1/L(t)$, pricing accuracy $\Sigma(t)^{1/2}$, and resiliency $\rho(t)$ are "microscopic" quantities, which are difficult to estimate.

Invariance relationships allow to infer microscopic quantities from macroscopic quantities ($\bar{C}_B$, m, and $\tau \cdot \theta^2$ are just constants).

Discussion

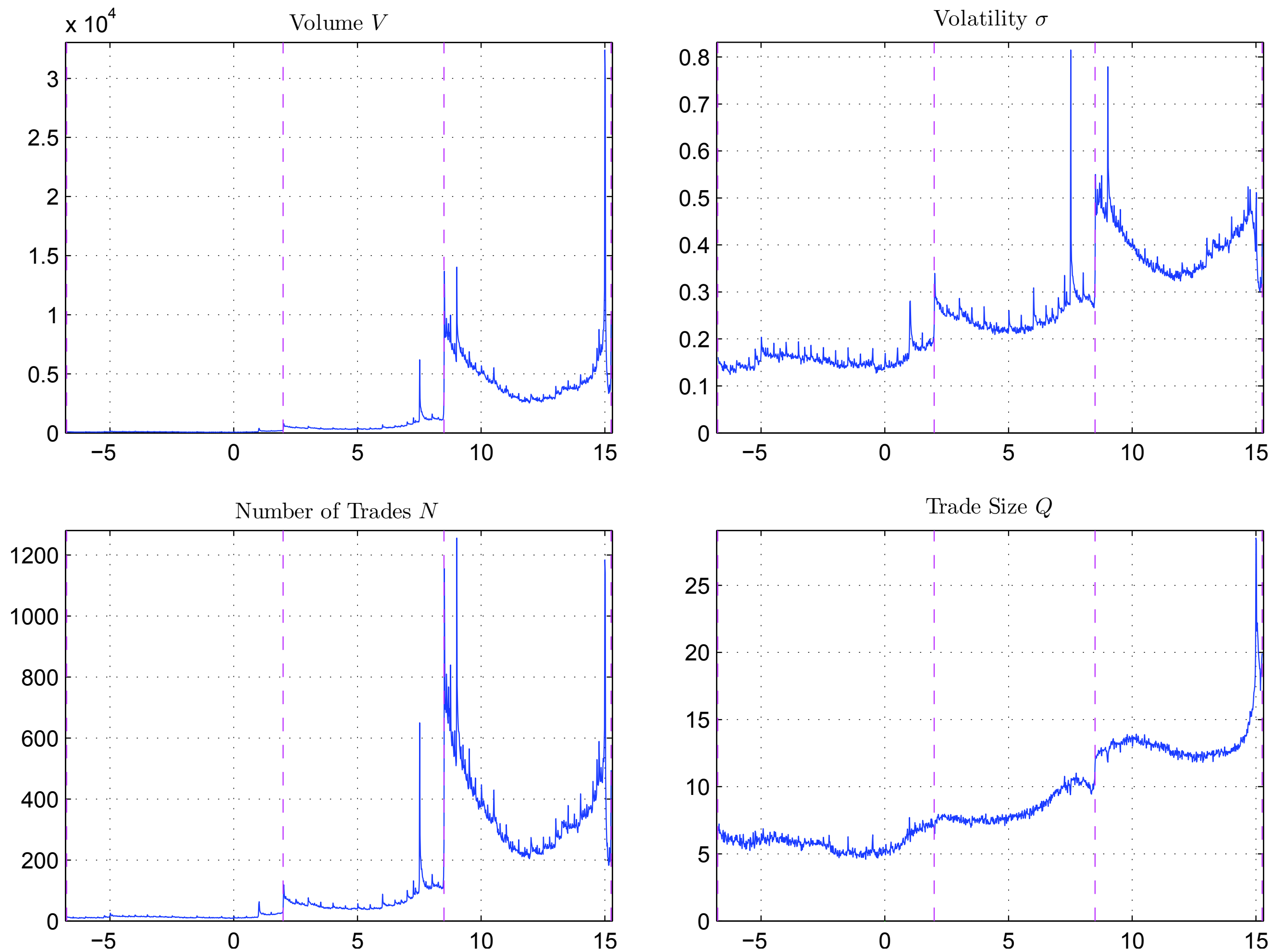
- The assumption that distinct bets result from distinct pieces of private information implies a particular level of granularity for both signals and bets.
- The invariance of bet sizes and their cost rely on the assumption that cost of a private signal c_I and the shape of the distribution of signals m are constant (c_I can be replaced by productivity-adjusted wage of a finance professional).
- The invariance of pricing accuracy and resiliency requires stronger assumptions: the informativeness of a bet $\tau \cdot \theta^2$ is constant.
- The model is motivated by the time series properties of a single stock as its market capitalization changes, but it can apply cross-sectionally across different securities.

Robustness of Assumptions

- Our structural model makes numerous restrictive assumptions. The empirical results we are about to describe are not consistent with the "linear-normal" assumptions of the model.
- The size of unsigned bets closely fits a log-normal distribution, not a normal distribution. A linear price impact model predicts market impact costs reasonably well, but a square root model of price predicts impact costs better.
- We conjecture that it should be possible to modify our structural model to accommodate those issues.
- The model is to be interpreted as a "proof of concept" consistent with the interpretation that the invariance hypotheses might apply more generally.

Intraday Patterns for S&P500 E-mini Futures

Intraday patterns for volume, volatility, number of trades, and average trade size (Andersen, Bondarenko, Kyle, Obizhaeva (2014))



Intraday Patterns for S&P500 E-mini Futures

Log of number of trades on log trading activity, by day and one-minute time interval (2008-2011). Predicted coeff. is 2/3. The fitted line is $\ln(N_{dt}) = -3.7415 + \mathbf{0.661} \cdot \ln(V_{dt} \cdot P_{dt} \cdot \sigma_{dt})$.