Robustness of equilibrium in the Kyle model of informed speculation

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HSE November 18, 2011

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Overview

- Motivation
- Related Literature
- Our Approach to Robustness
- Kyle (1983) Model
- Example: Linear Perturbations in Kyle (1985)
- Main Results
- Conclusion

Motivation: What is Robustness?

- Each strategic market participant makes assessments of
  - fundamentals
  - other agents' strategies

- In complex speculative trades, agents's beliefs about the strategies of others may be mis-specified.

- *Robustness* (to ambiguity): Small errors in agent's beliefs about other agents' trading strategies do not affect her expected payoffs (no first-order effects.)

Related Literature

1. Robustness to ambiguity of *beliefs*

   Stauber (2011)

   Initial Strategies and Beliefs – Bayesian Nash (BNE)
   "Perturbation" of beliefs – Ambiguous Beliefs (class of games)
   The initial BNE strategies are "approximately" optimal ($\epsilon-$ slack)

   Bewley (2002)
   Knightian decision theory; choice under incomplete preferences (unanimity rule)

2. Uncertainty about *payoffs* – body of literature (*finite action space*)

   Kajii and Morris (1997)
   Carlsson and van Damme (1993)

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- We consider a Kyle (1983) model

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  - More demanding than "$\epsilon-$slack"; more like "higher order slack"

Definition

The functional differential of the price with respect to the strategy $X(\cdot)$ is

$$\delta_X \overline{P}(x; X(\cdot), \delta X(\cdot)) = \lim_{\varepsilon \to 0} \left\{ \frac{\overline{P}(x; X(\cdot) + \varepsilon \delta X(\cdot)) - \overline{P}(x; X(\cdot))}{\varepsilon} \right\},$$

provided that the limit exists for every $\delta X(\cdot)$ (from the same functional space), and defines a functional, linear and bounded in $\delta X(\cdot)$

- Useful tool for our analysis
- Can be viewed as an extension of the *directional derivative* of functions depending on several variables

- Standard linear equilibrium of Kyle (1983) model of strategic trading is robust to conjecture errors
- If a non-linear equilibrium exists, then it is not robust

Kyle (1983) Model

Brief Description

- Single risk neutral informed trader
- Privately observes risky asset's liquidation value $v \sim N(0,1)$
- Liquidity traders trade a quantity $u \sim N(0,1)$ (market orders)
- The informed trader's strategy $X(\cdot)$ that details for each value of v, the traded quantity $x = X(v)$ (market order)
- Market makers $J \geq 3$ risk-neutral, profit-maximizing
- Each market maker $k = 1, \ldots J$ submits a limit order described by a non-discriminatory supply schedule $y_k(P)$
- The equilibrium price clears the market, $Y(P) = \sum_{k=1}^{J} y_k(P) = x + u$
- Obtain Kyle (1985) setting in the limit $J \to \infty$

Kyle (1983) Model (contd)

Standard linear equilibrium

- We focus on *symmetric* Nash equilibria
- Insider optimizes

$$\Pi_I\left(v, x; P_I\left(\cdot\right)\right) = E_u\left[\left(v - P_I\left(x+u\right)\right)x\right]$$

- Each market maker optimizes

$$\Pi_{M,k}\left(y_k, P; y_{M_k}\left(\cdot\right), X_{M,k}\left(\cdot\right)\right) = E_v\left[y_k(P-v); y_{M_k}\left(\cdot\right), X_{M,k}\left(\cdot\right)\right]$$

- Standard linear equilibrium

$$X^*(v) = \left(\frac{J-2}{J-1}\right)v, \quad \text{and} \quad P^*(Y) = \frac{1}{2}\left(\frac{J-1}{J-2}\right)Y.$$

Kyle (1983) Model (contd)

Nonlinear extension

- Define reaction functions, then match at equilibrium
- Insider's problem

Proposition

The first-order condition describing the insider's strategy is

$$v = \overline{P}_I \left(X \left(v \right) \right) + X \left(v \right) \overline{P}'_I \left(X \left(v \right) \right),$$

which must hold pointwise for each v.

- MMs' problem

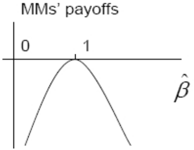
Proposition

The first-order condition for market maker k's problem is

$$y_k \left(P \right) = \left(J - 1 \right) y'_{M_k} \left(P \right) \left(P - P_e \left(Y; X_{M,k} \left(\cdot \right) \right) \right).$$

Insider's strategy: $X(v) = \beta v$
MMs' conjecture $\widehat{\beta}$ instead of β
Pricing rule (reaction to the OF $y = X(v) + u$):
$P(y) = \lambda y$ with $\lambda = \frac{\widehat{\beta}}{1+\widehat{\beta}^2}$

- **Insider's expected payoff:** $\overline{\pi}_I = E[\beta v (v - \lambda y)] = \beta(1 - \lambda\beta)$
- **MMs'expected payoff**: $\overline{\pi}_M = \lambda - \beta(1 - \lambda\beta) = \frac{\widehat{\beta}}{1+\widehat{\beta}^2}(1 + \beta^2) - \beta$
- **Equilibrium:** $\beta = \beta^* = 1$ **and** $\overline{\pi}_M = -\frac{(1-\widehat{\beta})^2}{1+\widehat{\beta}^2}$



Main Results II

Nonlinear perturbations, nonlinear equilibrium

- **MM k**: conjectures $X_{M_k}(\cdot)$
- Suppose we found Nash Equilibrium (NE)
- Consider a small variation in the conjecture of market maker k:

$$X_{M_k}(\cdot) = X^*(\cdot) + \delta X_{M_k}(\cdot).$$

- Small deviation from the NE
- Same with other MMs and Insider
- Our main results are summarized by the following:

Theorem

1. *The standard linear equilibrium of Kyle (1983) is robust with respect to small conjecture errors of the market makers or the informed trader.*
2. *The only equilibrium of Kyle (1983) that is robust in the sense of Definition 1 is the standard linear equilibrium.*

Main Results III

Robustness of linear equilibrium

- We establish an even stronger robustness notion:

Proposition

In the standard linear Nash equilibrium of Kyle (1983), the first variations of all agent's expected payoffs with respect to variations in the conjectures of any agent vanishes.

- This robustness result is stronger than the notion defined in Definition 1, because it says that in a standard linear equilibrium, each market participant is indifferent to small errors in his or her own beliefs, *and* to errors in the beliefs that others hold

Conclusion

- We establish a strong sense in which the standard linear Nash equilibrium of Kyle (1983, 1985) model is robust
- We prove that each market participant is indifferent to small errors in his or her own beliefs and to small errors in the beliefs that others hold
- We prove that the only robust Nash equilibrium of Kyle (1983) model is the standard linear one
- The notion of robustness that we establish is appealing: action spaces are *functional* and the strategic interactions are especially complex