Affine Option Pricing Model in Discrete Time

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Introduction

Continuous vs Discrete

Continuous time affine models with stochastic volatility:

- Cox, Ingersoll, and Ross (1985, Econometrica)
- Heston (1993, RFS)
- Duffie, Pan, and Singleton (2000, Econometrica)

Continuous vs Discrete

Popular because of computational convenience ...

... with historical probability measure (P):

- Robustness to temporal aggregation:
  Meddahi and Renault (2004, JoE)
- Robustness to cross-sectional aggregation (portfolio)

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... with risk neutral probability measure (Q):

- Structure preserving change of measure
  (affine structure preserved)
- Analytical tractability of computing derivative prices
  (inverse Fourier transform)

Discrete Time Extension

Volatility model:

- Darolles, Gourieroux, and Jasiak (2006, JTSA)
- Gourieroux and Jasiak (2006, JoF)

Option pricing model with conditional skewness:

- Feunou and Tedongap (2012, JBES)

Discrete Time Extension

Advantages of discrete time:

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Challenges of discrete time:

- Accommodating leverage effect
- Keeping the advantage of structure preserving change of measure historical/risk-neutral

Affine Stochastic Volatility Model

The Model

CAR Volatility: Darolles, Gourieroux, and Jasiak (2006, JTSA)

$$E\left[\exp\left\{-u\sigma_{t+1}^2\right\} \middle| I_t\right] = \exp\left\{-a(u)\,\sigma_t^2 - b(u)\right\}$$

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Log Excess Return

$$E\left[\exp\left\{-v r_{t+1}\right\} \middle| I_t \cup \sigma_{t+1}^2\right] = \exp\left\{-\alpha(v)\,\sigma_{t+1}^2 - \beta(v)\,\sigma_t^2 - \gamma(v)\right\}$$

The Model

Joint Return and Volatility

$$E\left[\exp\left\{-u\sigma_{t+1}^2 - vr_{t+1}\right\}\middle| I_t\right] = \exp\left\{-l\left(u,v\right)\sigma_t^2 - g\left(u,v\right)\right\}$$

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$$\begin{aligned} l(u,v) &= a[u+\alpha(v)]+\beta(v) \\ g(u,v) &= b[u+\alpha(v)]+\gamma(v) \end{aligned}$$

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$\alpha(v) \neq 0 \iff$ leverage!

Risk-Neutral Distribution

The affine structure is kept from P to Q

when

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Stochastic Discount Factor (SDF):

$$M_{t,t+1}(\theta) = \exp(-r_{f,t}) \exp\left\{ m_0(\theta) + m_1(\theta)\sigma_t^2 - \theta_1 \sigma_{t+1}^2 - \theta_2 r_{t+1} \right\}$$

- risk prices $\theta_1 \leq 0$ and $\theta_2 \geq 0$
- $m_0(\theta)$, $m_1(\theta)$: bonds and stocks are priced correctly

Risk-Neutral Distribution

Risk-neutral pricing:

$$E^{Q}\left[H\left(r_{t+1},\sigma_{t+1}^{2}\right)\middle| I_{t}\right]=\exp\left(r_{f,t}\right)E\left[M_{t,t+1}\left(\theta\right)H\left(r_{t+1},\sigma_{t+1}^{2}\right)\middle| I_{t}\right]$$

for any function H

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Risk-neutral distribution:

$$E^Q\left[\exp\left(-u\sigma^2_{t+1} - vr_{t+1}\right)\middle| I_t\right] = \exp\left(-I^*\left(u,v\right)\sigma^2_t - g^*\left(u,v\right)\right)$$

with

$$\begin{aligned} I^*\left(u,v\right) &= I\left(\theta_1+u,\theta_2+v\right) - I\left(\theta_1,\theta_2\right) \\ g^*\left(u,v\right) &= g\left(\theta_1+u,\theta_2+v\right) - g\left(\theta_1,\theta_2\right) \end{aligned}$$

Affine Moments

Volatility moments:

$$
\begin{aligned}
E\left[\sigma_{t+1}^2 \middle| I_t\right] &= a'(0)\,\sigma_t^2 + b'(0) \\
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Leverage effect:

$$\phi \approx Corr\left[r_{t+1}, \sigma_{t+1}^2 \middle| I_t\right] = \alpha'(0)\left(\frac{V\left[\sigma_{t+1}^2 \middle| I_t\right]}{V\left[r_{t+1} \middle| I_t\right]}\right)^{1/2}$$

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Option Pricing

Generalized Black-Scholes

Assume $\alpha(v), \beta(v), \gamma(v)$ are quadratic, then

$$r_{t+1} | I_t^{\sigma} \sim N(E[r_{t+1} | I_t^{\sigma}], V[r_{t+1} | I_t^{\sigma}])$$

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Option price:

$$C_t\left(x_t, \phi\right) = E_t^Q\left[BS\left(S_t \xi_{t,t+1}(\phi), \left(1 - \phi^2\right)\sigma_{t+1}^2, K\right)\right]$$

where $x_t = \log\left(K / S_t\right)$ is the moneyness and

$$\log \xi_{t,t+1}(\phi) = E^Q\left[r_{t+1} | I_t^{\sigma}\right] + \frac{1}{2} V^Q\left[r_{t+1} | I_t^{\sigma}\right]$$

is price distortion

Leverage and Volatility Smirk

Two effects of ϕ:

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Around $\phi = 0$ the first order effect is through volatility:

$$C_t(x_t,\phi) \approx C_t(x_t,0) + k\phi \cdot Cov^Q\left[\sigma_{t+1}^2, \Phi(d) \middle| I_t\right]$$

with

$$d = \frac{1}{2}\sigma_{t+1} - \frac{x_t}{\sigma_{t+1}}$$

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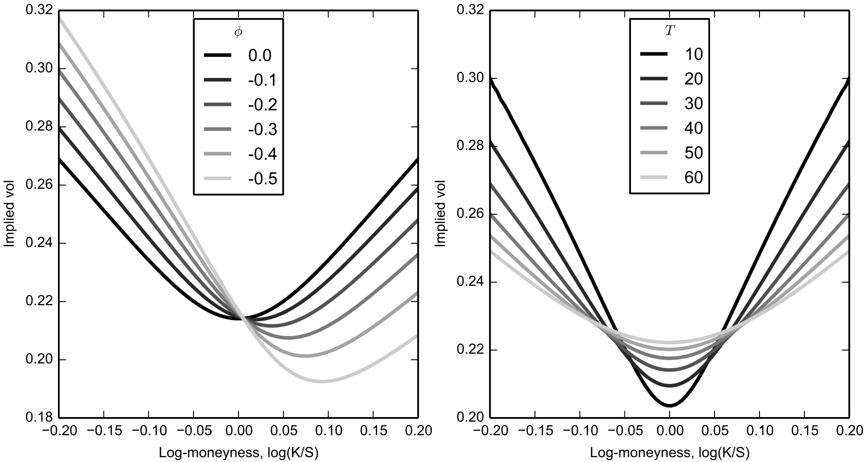
Cov() more positive out of the money

$\Longrightarrow$ the smile is pushed down on the out of the money side

Leverage and Volatility Smirk



Estimation

Maximum Likelihood

Joint likelihood

$$
\begin{aligned}
f\left(r_{t+1}, \sigma_{t+1}^2 \,\middle|\, \sigma_t^2; c, \rho, \delta, \phi, \theta_2\right) = & f\left(r_{t+1} \,\middle|\, \sigma_{t+1}^2, \sigma_t^2; \phi, \theta_2\right) \\
& \times f\left(\sigma_{t+1}^2 \,\middle|\, \sigma_t^2; c, \rho, \delta\right)
\end{aligned}
$$

Maximum Likelihood

Joint likelihood

$$f\left(r_{t+1},\sigma^2_{t+1}\middle|\sigma^2_t;c,\rho,\delta,\phi,\theta_2\right)=f\left(r_{t+1}\middle|\sigma^2_{t+1},\sigma^2_t;\phi,\theta_2\right)\times f\left(\sigma^2_{t+1}\middle|\sigma^2_t;c,\rho,\delta\right)$$

where

$$\begin{aligned} f\left(r_{t+1}\middle|\sigma^2_{t+1},\sigma^2_t;\phi,\theta_2\right) &\sim Normal \\ f\left(\sigma^2_{t+1}\middle|\sigma^2_t;c,\rho,\delta\right) &\sim nc-Gamma \end{aligned}$$

σ^2_{t+1} is ARG(1) from Gourieroux and Jasiak (2006, JoF)

Spectral GMM

Singleton (2001, JoE), Chacko and Viceira (2003, JoE)

Moment functions:

$$g_t(u,\theta) = Z_t \cdot \begin{bmatrix} \exp\left\{-u\sigma_{t+1}^2\right\} - \exp\left\{-a(u)\,\sigma_t^2 - b(u)\right\} \\ \exp\left\{-ur_{t+1}\right\} - \exp\left\{-\alpha(u)\,\sigma_{t+1}^2 - \beta(u)\,\sigma_t^2 - \gamma(u)\right\} \end{bmatrix}$$

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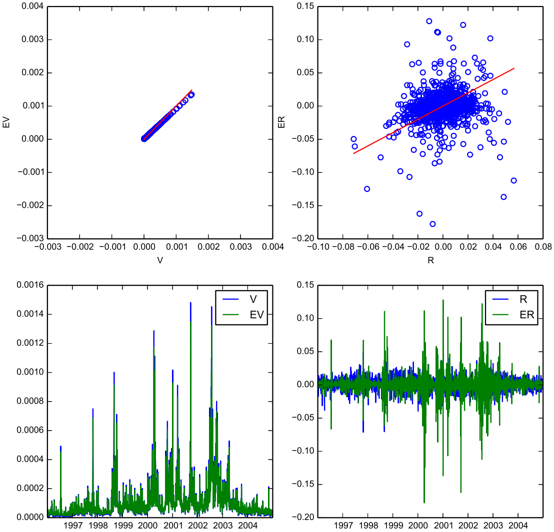
Moments to match:

$$E\begin{bmatrix} \text{Re}\left\{g_t(u,\theta)\right\} \\ \text{Im}\left\{g_t(u,\theta)\right\} \end{bmatrix} = 0$$

Model Fit



Parameter Estimates

	MLE		GMM	
	$\hat{\theta}$	t	$\hat{\theta}$	t
c	**2.4e-5**	[29.9]	**6.7e-6**	[4.5]
ρ	**0.66**	[39.9]	**0.91**	[28.5]
δ	**1.45**	[29.5]	**1.18**	[6.4]
ϕ	**-0.21**	[-14.3]	**-0.22**	[-10.2]
θ_2	**1.57**	[0.7]	**1.90**	[0.9]

Vol Risk Price Calibration

$$\hat{\theta}_1 = \arg\min_{\theta_1} RMSE_{IV}(\theta_1) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(IV_j^{Market} - IV_j^{Model}(\theta_1)\right)^2}$$

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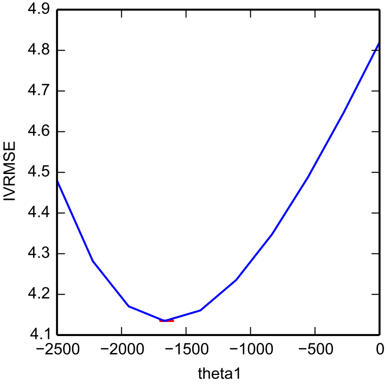

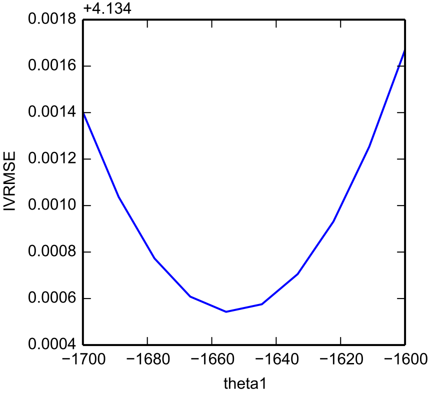

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  - More flexibility for higher order moments
- Work in progress: take advantage of this flexibility for empirical fit better than standard Heston:
  - Two volatility factors (slow and fast mean reverting)
  - Mixture component in return r_{t+1} given I_t^σ for more kurtosis (gamma mixture to keep the affine structure) reminiscent of jumps in continuous time

Thank you!

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