Comparison among Portfolio Managers and Asset Specialization by S. Basak and D. Makarov

Discussion by Mihail Zervos

- Consider a market with a risk-free asset and two risky assets with price processes S_1, S_2, given by

$$dS_{it} = (r + \mu_i)S_{it}\,dt + \sigma_i S_{it}\,d\omega_{it}, \tag{1}$$

  where ω_1, ω_2 are standard one-dimensional Brownian motions with correlation $0 < \varrho < 1$.

- Consider two managers who invest according to one of the following scenarios:

  - (*asset specialisation*) Manager i is constrained to invest in the risk-free asset and in risky asset i only.

  - (*no asset specialisation*) Both managers invest in all three assets.

  - (*partial asset specialisation*) One of the managers is constrained to asset specialisation, while the other one has no asset specialisation.

- In any of the cases,

$$\text{manager 1 wants} \quad \frac{1}{1-\gamma_1}\mathbb{E}\left[\left(\frac{W_{1T}}{W_{2T}^{\vartheta_1}}\right)^{1-\gamma_1}\right] \quad \text{maximised,} \tag{2}$$

$$\text{while, manager 2 wants} \quad \frac{1}{1-\gamma_2}\mathbb{E}\left[\left(\frac{W_{2T}}{W_{1T}^{\vartheta_2}}\right)^{1-\gamma_2}\right] \quad \text{maximised} \tag{3}$$

over all their permissible investment strategies, which gives rise to a game.

Here, the constants $\gamma_i > 0$ and $0 \leq \vartheta_i \leq 1$ capture the managers' *attitude towards risk* and *relative performance bias*.

- Suppose that manager i is constrained to invest in the risk-free asset and in risky asset i only, and denote by φ_{it} the fraction of wealth that manager i has invested in risky asset i at time t.

  An equilibrium strategy of the resulting non-zero sum game is a pair $(\varphi_1^*, \varphi_2^*)$ such that

$$\tilde{v}_{1T}(\varphi_1, \varphi_2^*) \leq \tilde{v}_{1T}(\varphi_1^*, \varphi_2^*) \quad \text{and} \quad \tilde{v}_{2T}(\varphi_1^*, \varphi_2) \leq \tilde{v}_{2T}(\varphi_1^*, \varphi_2^*) \quad \text{for all } \varphi_1, \varphi_2. \tag{4}$$

  where

$$\tilde{v}_{1T}(\varphi_1, \varphi_2) = \frac{1}{1-\gamma_1}\mathbb{E}\left[\left(\frac{W_{1T}}{W_{2T}^{\vartheta_1}}\right)^{1-\gamma_1}\right] \text{ and } \tilde{v}_{2T}(\varphi_1, \varphi_2) = \frac{1}{1-\gamma_2}\mathbb{E}\left[\left(\frac{W_{2T}}{W_{1T}^{\vartheta_2}}\right)^{1-\gamma_2}\right]. \tag{5}$$

- The paper derives a unique Markovian equilibrium strategy for each of the three types of games (i.e., with / without / with partial asset specialisation).

- If they follow the equilibrium strategies, the two managers receive the expected payoffs

$$J_1^{Sp}(W_{10}, W_{20}) = \frac{k_1^{Sp}}{1-\gamma_1}\left(\frac{W_{10}}{W_{20}^{\vartheta_1}}\right)^{1-\gamma_1} \quad \text{and} \quad J_2^{Sp}(W_{10}, W_{20}) = \frac{k_2^{Sp}}{1-\gamma_2}\left(\frac{W_{20}}{W_{10}^{\vartheta_2}}\right)^{1-\gamma_2} \tag{6}$$

in the case of asset specialisation,

$$J_1^{NoSp}(W_{10}, W_{20}) = \frac{k_1^{NoSp}}{1-\gamma_1}\left(\frac{W_{10}}{W_{20}^{\vartheta_1}}\right)^{1-\gamma_1} \quad \text{and} \quad J_2^{NoSp}(W_{10}, W_{20}) = \frac{k_2^{NoSp}}{1-\gamma_2}\left(\frac{W_{20}}{W_{10}^{\vartheta_2}}\right)^{1-\gamma_2} \tag{7}$$

in the case of no asset specialisation, and, e.g.,

$$J_1^{1\,NoSp\,/\,2\,Sp}(W_{10}, W_{20}) \quad \text{and} \quad J_2^{1\,NoSp\,/\,2\,Sp}(W_{10}, W_{20}) \tag{8}$$

in the case where manager 1 does not specialise and manager 2 specialises.

- Suppose that a switch of manager i from specialisation to no specialisation incurs a cost of $\lambda_i W_{i0}$ (if $\lambda_i > 0$) or receives a payoff $\lambda_i W_{i0}$ (if $\lambda_i < 0$) at time 0.

  The effect of both managers switching from specialisation to no specialisation should involve the comparison of the expected payoffs

$$J_1^{Sp}\big(W_{10}, W_{20}\big) \quad \text{and} \quad J_1^{NoSp}\big((1-\lambda_1)W_{10}, (1-\lambda_2)W_{20}\big) \tag{9}$$

  as well as

$$J_2^{Sp}\big(W_{10}, W_{20}\big) \quad \text{and} \quad J_2^{NoSp}\big((1-\lambda_1)W_{10}, (1-\lambda_2)W_{20}\big) \tag{10}$$

  Similarly, the effect of manager 1 switching from specialisation to no specialisation but manager 2 sticking to specialisation should involve the comparison of the expected payoffs

$$J_1^{Sp}\big(W_{10}, W_{20}\big) \quad \text{and} \quad J_1^{1\,NoSp\,/\,2\,Sp}\big((1-\lambda_1)W_{10}, W_{20}\big) \tag{11}$$

  as well as

$$J_2^{Sp}\big(W_{10}, W_{20}\big) \quad \text{and} \quad J_2^{1\,NoSp\,/\,2\,Sp}\big((1-\lambda_1)W_{10}, W_{20}\big) \tag{12}$$

  Of course, there are several other similar possibilities.

- Overall, I very much liked the problem and the issues that the paper addresses!
- I believe that the paper could have been written much better. In particular,

  - the various issues analysed could have been formulated with more care,

  - the notation could have been clearer,

  - the dependence of various quantities on crucial arguments could have been more carefully considered and streamlined (e.g., my impression is that Proposition 2 compares $J_1^{Sp}\big(W_{10}, W_{20}\big)$ with $J_1^{NoSp}\big((1-\lambda_1)W_{10}, W_{20}\big)$).
- My impression is that the possibility of switching at time 0 only is associated with multiple equilibria. The possibility for allowing dynamic switching might result in a unique equilibrium. However, this might be a very hard problem...
- The paper considers Markovian strategies only. An appropriate verification theorem can be developed to show that the equilibrium strategies derived are unique in a much wider class of strategies.
- Assumptions such as, e.g., that manager 1 knows manager 2's ϑ_2 and γ_2 are rather unrealistic. However, this is justified because very little theory has been developed on the type of problem that the paper analyses.