Accounting Rules, Equity Valuation, and Growth Options*

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Abstract

In a model with irreversible capacity investments, we show that financial statements prepared under replacement cost accounting provide investors with sufficient information for equity valuation purposes. Under alternative accounting rules, including historical cost and value in use accounting, investors will generally not be able to value precisely a firm's growth options and, therefore, its equity. For these accounting rules, we describe the range of valuations that is consistent with the firm's financial statements. We further show that replacement cost accounting preserves all value-relevant information if the firm's investments are reversible. However, the directional relation between the firm's equity value and the replacement cost of its assets is different from that in the setting with irreversible investments.

1 Introduction

The FASB Conceptual Framework for Financial Reporting (FASB 2010) states that "the objective of general purpose financial reporting is to provide financial information about the reporting entity that is useful to existing and potential investors, lenders, and other creditors in making decisions about providing resources to the entity." To achieve this objective, it is important to understand the informational needs of different groups of financial statements users, including the current and potential future shareholders of the firm. However, the existing theoretical literature on accounting-based equity valuation provides little guidance on the relative desirability of alternative accounting rules (see, e.g., Ohlson 1995, Feltham and Ohlson 1995).[1] In a model without uncertainty, Nezlobin (2012) shows that replacement cost accounting is essentially the only depreciation policy under which there exists a mapping between a firm's current accounting data and its equity value.[2] Our goal is to study how the resolution of uncertainty regarding the firm's investments should be reflected in its financial statements to provide information useful to equity investors.

We employ the real options framework to model the problem of accounting based valuation under uncertainty. Specifically, we consider a firm that makes repeated investments in capital goods and uses the resulting capital stock for production. The firm's investments are irreversible - once purchased and put in place (or constructed), the firm's capital goods cannot be sold. In practice, investments can be fully or partially irreversible because they are firm-specific (e.g., highly specialized equipment), industry-specific, or because the market for used capital goods is affected by the "lemons" problem (see, e.g., Dixit and Pindyck, 1994, p.8).[3] Pindyck (1988) characterizes the firm's optimal investment policy under the

[1] Most commonly cited valuation models, such as Ohlson (1995), Feltham and Ohlson (1995), and Ohlson and Juettner-Nauroth (2005), express the value of a firm as a function of accounting variables, assuming an exogenous process for firm's residual earnings. These papers do not specify the accounting rules that need to be applied in different economic environments to generate a residual earnings process conforming to the assumed specification. For example, one of the main questions raised by Penman (2005) in his discussion of Ohlson and Juettner-Nauroth (2005) was "Where's the accounting?" Several papers explicitly model accounting rules but treat both investment and cash flow processes as exogenous, see, e.g., Feltham and Ohlson (1996), Ohlson and Zhang (1998) and X. Zhang (2000).

[2] In the model of Nezlobin (2012), there is no uncertainty about the price of capital goods or their productivity, so replacement cost accounting is effectively a fixed depreciation schedule. This depreciation policy was first identified in Rogerson (2008) and also studied in, e.g., Rajan and Reichelstein (2009) and Nezlobin et al. (2012).

[3] We initially focus on irreversible investments because illiquid assets present arguably the most difficult scenario for designing an accounting treatment that conveys all value relevant information to investors. We later consider a setting with reversible investments.

assumption that demand for its product is stochastic. Since the focus of our paper is on the accounting treatment of uncertainty related to capital investments, we extend the model of Pindyck (1988) and Dixit and Pindyck (1994, Chapter 11) to allow for (i) a stochastic rate of physical depreciation of the firm's assets in place and (ii) a stochastic price of new capital goods. The three stochastic processes that determine the firm's economic environment - demand for the firm's output, the physical productivity of its assets in place, and the price of new capital goods - are allowed to be mutually correlated.

We assume that equity investors know the main parameters of the firm's economic environment, e.g., the mean and variance of the growth rate of the output market size. However, they do not observe the realizations of the underlying stochastic processes, e.g., the actual size of the output market. In valuing the firm's equity investors use the firm's financial statements that provide information about its assets in place and the cash flows they generate. Our objectives are twofold. First, we identify accounting rules with the property that the firm's equity value can be precisely determined based on its accounting information. Second, for commonly considered accounting rules without this property, we seek to characterize the range of valuations consistent with the firm's financial statements.

We first show that the optimal investment policy in our model is of a barrier control type.[4] Specifically, the firm invests only when the ratio of its operating cash flow to the replacement cost of its capital stock exceeds a certain threshold. We refer to this ratio as the cash return on economic assets or, simply, cash return on assets when there is no room for confusion.[5] The firm's investment increases both the numerator and the denominator of this ratio. Under the standard assumption that the marginal revenue with respect to capital is decreasing in capital stock, the overall ratio decreases as a result. The firm chooses its investment so as to return the ratio of operating cash flows to the replacement cost of assets to a level just below the investment threshold. Consistent with much of the real options literature, the NPV of the marginal project at the investment threshold is strictly greater than zero: the firm optimally takes into account the fact that the investment can be postponed if market conditions improve but cannot be undone if they deteriorate. The NPV of the marginal project is thus equal to the opportunity cost of the option to wait.

[4]Dixit and Pindyck (1994, p. 362) discuss the optimality of a barrier-control policy in a model where the price of capital goods and the physical depreciation rate of assets do not change over time.

[5]In our model, the firm is all-equity financed, there are no taxes and no accruals other than the ones related to capital assets (depreciation and revaluations). Therefore, the operating cash flow is essentially equal to the firm's earnings before interest, taxes, depreciation and amortization (EBITDA).

It follows from the above discussion that the replacement cost of assets in place plays an important role in determining the optimal investment policy of the firm. The replacement cost of assets can be viewed as the amount that the firm would have to pay at the current price of new capital goods to replicate the productive capacity of its existing capital stock. Alternatively, the replacement cost can be calculated for each vintage of capital goods as the depreciated historical cost of that vintage (reflecting all shocks to the productivity of assets incurred up to the current date) multiplied by the ratio of the price of new assets today to the price of new assets at the time of purchase. We refer to an accounting system that sets the book value of assets equal to their replacement cost as replacement cost accounting. Our definition of replacement cost accounting is consistent with the cost approach to fair value measurement as defined in IFRS 13.[6]

The model allows us to derive an analytical expression for the firm's equity value. Specifically, the firm's value is equal to the sum of two components: the present value of cash flows to be generated by existing assets and the value of the firm's capacity expansion (growth) options.[7] The value of cash flows from existing assets can be calculated by multiplying the current operating cash flow by a capitalization factor that reflects the firm's cost of capital and other parameters of the the firm's economic environment, such as the expected growth in demand for firm's output and the drift and variance of the asset productivity process. The value of the capacity expansion options is proportional to the present value of cash flows from existing assets with the coefficient of proportionality being determined by how far the current cash return on assets ratio is from the investment threshold. It follows that if financial statements are prepared using replacement cost accounting, investors will have sufficient information to value the firm precisely at each point in time. It is useful to recall that the firm's capital goods cannot be sold in our model, so investments in those assets are sunk once they are incurred. It is, however, still important for investors to know the replacement cost of the firm's assets to be able to value the firm's growth options.

Next, we explore the informational properties of alternative accounting rules. We start by assuming that the firm prepares financial statements on a cash basis, i.e., investors observe

[6] IFRS 13 defines the cost approach as "a valuation technique that reflects the amount that would be required currently to replace the service capacity of an asset (often referred to as current replacement cost)." IAS 16 allows to carry property, plant and equipment at their fair value determined in accordance with IFRS 13 (see IAS 16, Revaluation Model, paragraph 31).

[7] Similar decompositions have been obtained in other settings; see, for instance, Lindenberg and Ross (1981), Berk et al. (1999), Abel and Eberly (2011), Kogan and Papanikolaou (2014).

only the cash flows from operating and investment activities. While cash flow information alone is not sufficient for valuation purposes in all states of the world, investors can (i) calculate the firm's value precisely when *new* investment is observed, and (ii) always calculate upper and lower bounds on the firm's equity value.[8]

In the investment region, when the firm is expanding its capacity, the replacement cost of assets in place does not provide incremental value-relevant information relative to the firm's operating cash flow. The reason for this is that the mere fact that the firm is exercising its expansion options tells investors that the cash return on assets ratio is at the investment threshold. This information is then sufficient to value the firm conditional on the value of the operating cash flow. In the inaction region (i.e., when cash return on assets is below the investment threshold), the firm's expansion options cannot be valued precisely without knowing the replacement cost of assets in place. However, their value can be bounded from below by zero and from above by the value of growth options assuming the firm had been at the investment threshold.[9]

These results demonstrate that while financial statements prepared on a cash basis provide useful information to investors, they are not sufficient to accurately value the firm's growth options in the inaction region. We further show that the bounds on the firm's value obtained under cash accounting cannot be improved if financial statements are prepared using *value in use* accounting (where the book value of assets at each date is set equal to the present value of cash flows that they are expected to generate) or *historical cost* accounting (assets are carried at historical cost with depreciation reflecting the physical shocks to productivity). The reason for this is that the value of the firm's capacity expansion options depends on the current price of new capital goods, and the changes in the price of new capital goods do not get reflected in the financial statements under historical cost or value in use accounting if the firm is in the inaction region.

Next, we model a *conditionally conservative* accounting system, under which the book value of assets is written down immediately if it exceeds their replacement cost, but is not written up if the replacement cost is higher.[10] We show that financial statements prepared

[8]Cash flow information alone would be sufficient for precise equity valuation if both the economic depreciation of existing assets and the price of new ones were not stochastic.

[9]Accordingly, the firm's equity value is bounded by the present value of cash flows from existing assets and by the value of the firm had it been at the investment threshold.

[10]Here, we consider a conditionally conservative system based on replacement cost accounting. In practice, the standards for asset impairments are different under U.S. GAAP and IFRS, and the write-down amounts

under such rules allow investors to improve (relative to cash basis accounting) the *upper* bound on the value of the firm's growth options and, therefore, on its equity value. If accounting is conservative, then the book value of assets always understates their replacement cost. Therefore, the *reported* cash return on assets, i.e., the ratio of operating cash flow to the book value of assets, can serve as an upper bound for the cash return on economic assets. The value of capacity expansion options is monotonically increasing in the cash return on economic assets, so this value can be more precisely bounded from above if accounting is conditionally conservative.

We also consider a scenario where the firm's investments are reversible: the firm's capital goods can be sold at the price of new capital goods with equivalent productive capacity.[11] In this setting, the value of the firm's equity is also shown to depend on the current replacement cost of the firm's capital stock. However, the direction of the relation between the firm's equity value and the replacement cost of its assets changes to the opposite: with reversible investments, the replacement cost of assets is positively associated with the equity value. When the firm cannot revise its capital stock downwards, a high replacement cost of assets indicates that the value of growth options is low (new assets are too expensive or the firm has too much capacity already). On the other hand, if investments are reversible, then the firm would keep assets with a high replacement cost only if the present value of cash flows to be generated by those assets is even higher. Our results show that while knowing the replacement cost of assets is important for investors in both scenarios, the relation between this variable and the firm's equity value critically depends on whether the firm can sell its used capital goods.

Our paper is related to several strands of research in finance and accounting. First, our finding that investment is determined by the operating cash flow and replacement cost of assets in place is consistent with much of the theoretical and empirical literature on investment (see, e.g., Tobin 1969, Hayashi 1982, Gomes 2001, Abel and Eberly 2011).[12] We extend this literature by allowing for stochastic productivity of existing assets and a

are determined by different quantities, including the asset's value in use and its fair value. We discuss an application of our model to a setting where asset write-downs are recognized according to IAS 36 in Appendix B.

[11] For a model with time-dependent (but not stochastic) depreciation and reversible investments, see Abel and Eberly (2011).

[12] Most commonly used approaches of estimating Tobin's q reconstruct the book value of assets under replacement cost accounting from the firms' investment histories (see, e.g., Salinger and Summers 1983, Perfect and Wiles 1984, and Lewellen and Badrinath 1997).

stochastic price in the capital goods market. Our finding that replacement cost accounting provides useful information to investors is consistent with the empirical evidence in Gordon (2001).

Our paper also contributes to the growing accounting literature on real options. Several papers examined the problems of incentive provision and product pricing with optionality (see, e.g., Arya and Glover 2001, Pfeiffer and Schneider 2007, Johnson et al. 2013, Reichelstein and Rohlfing-Bastian 2015, Nezlobin et al. 2015). To the best of our knowledge, the only other theoretical paper that studies accounting-based equity valuation with real options is G. Zhang (2000).[13] In the model of that paper, the firm can expand, keep unchanged, or discontinue its operations depending on the market conditions at a certain date. In contrast to our paper, G. Zhang (2000) studies European rather American real options, i.e., the decisions cannot be postponed or advanced in his model. Another difference between our paper and his is that, under our assumptions, it is always optimal for the firm to continue its operations, while G. Zhang (2000) explicitly considers an abandonment option. Lastly, since the main focus of our paper is on characterizing accounting rules that provide information useful to investors, we introduce restrictions on the information set available to outsiders who are valuing the firm based on financial statements.

The rest of the paper is organized as follows. We describe the firm's transactions in Section 2, and the information that is available to investors under alternative accounting rules in Section 3. The firm's optimal investment policy and its equity value are characterized in Section 4. In Section 5, we consider a scenario with reversible investments. Section 6 concludes.

2 The Firm's Transactions

Our model builds on the continuous time capacity choice model of Dixit and Pindyck (1994, Chapter 11). Consider a firm that makes repeated investments in capital assets and uses the productive capacity of those assets to make its output. Let K_t denote the firm's physical capital stock at time t. Assume that the firm's operating cash flow at time t is given by

$$CF_t = X_t K_t^{\alpha}, \tag{1}$$

[13] For an empirical study based on G. Zhang (2000) model, see Hao et al. (2011).

where $0 < \alpha < 1$ is the elasticity of the operating cash flow to capital, and X_t is a shift parameter that reflects the current demand for the firm's output. The parameter α is less than one possibly reflecting diminishing physical returns to production or a downward-sloping demand curve for the firm's product.[14] We assume that the shift parameter X_t follows a geometric Brownian motion

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X dz_X, \tag{2}$$

where μ_X is the drift in demand for the firm's product and σ_X is the instantaneous variance of the demand process.

The firm can purchase capital goods instantaneously and frictionlessly at time t at a price of P_t per unit. The price of new capital goods follows a geometric Brownian motion with constant drift μ_P and constant instantaneous variance σ_P^2, so

$$\frac{dP_t}{P_t} = \mu_P dt + \sigma_P dz_P. \tag{3}$$

The drift in the price of new assets, μ_P, can reflect expected inflation (if $\mu_P > 0$) or the notion that new assets become more efficient over time due to technology improvements (if $\mu_P < 0$).

Let dI_t denote the physical quantity of capital goods that the firm purchases at time t; the corresponding cash outflow at time t (cash flow from investing activities) is then given by $dI_t \cdot P_t$.[15] We assume that the firm cannot adjust its capital stock downwards; i.e., the

[14] For example, the specification above obtains if one assumes (i) the standard Cobb-Douglas production technology, $q_t = K_t^s$ with $0 < s$, where q_t is the number of units of the output product the firm can make at time t, and (ii) constant elasticity demand curves:

$$P_t(q_t) = X_t \cdot q_t^{-\frac{1}{\eta}},$$

where $\eta > 1$ is the price elasticity of demand. Then, the total cash flow is given by:

$$CF_t = P_t(q_t) \cdot q_t = X_t K_t^{s\left(-\frac{1}{\eta}+1\right)}.$$

To ensure that the optimal production volume is always finite, we impose the requirement that $0 < s\left(-\frac{1}{\eta}+1\right) < 1$. This requirement is satisfied if the firm's production technology exhibits decreasing returns to scale ($s < 1$), or if returns to scale are constant or increasing ($s \geq 1$) but η is sufficiently small (i.e., demand is sufficiently inelastic). It is also straightforward to extend our results to a setting where the firm's production function requires a second input, labor, that can be purchased instantaneously after the firm observes the current demand.

[15] With this notation, I_t is the total quantity of capital goods installed from the firm's inception up to time t.

investments in capital goods are irreversible ($dI_t \geq 0$). The firm's capital stock evolves according to the following equation:

$$dK_t = -\delta K_t dt + \sigma_K K_t dz_K + dI_t, \tag{4}$$

The first two terms in the right-hand side of (4) model the economic depreciation of the firm's assets in place at time t. The physical rate of depreciation is stochastic with mean δ and instantaneous variance σ_K^2. The last term in the right-hand side of (4) reflects the installation of the newly purchased capital goods.

To make our problem as general as possible we allow for the Brownian motions dz_X, dz_P, and dz_K to be mutually correlated:

$$\rho_{XP} = \frac{dz_X dz_P}{dt},$$

$$\rho_{KP} = \frac{dz_K dz_P}{dt},$$

$$\rho_{XK} = \frac{dz_X dz_K}{dt}.$$

Therefore, our analysis allows for situations where the prices of inputs move together with the price of output (e.g., costs of constructing and maintaining drilling rigs can be correlated with the oil price).[16] The model also accommodates technology shocks that affect, to a varying degree, both the productivity of the firm's existing assets as well as the price of new ones. Lastly, it is also conceivable that shocks to demand in the output market are correlated with the productivity of the firm's assets: for example, disruptions in supply (caused by unexpectedly low productivity of assets) may negatively affect the current and future demand for the firm's product.

At time t, the firm observes its current capital stock, K_t, the price of new capital goods, P_t, and the current demand for its product, X_t and decides whether or not to purchase another dI_t units of new capital goods. The firm's investment policy, I_t, is therefore determined by the three state variables: K_t, X_t, and P_t. The firm chooses its investments to maximize the

[16]Generally, such situations may arise when prices in both product and capital goods markets are affected by common macroeconomic factors.

present value of its cash flows, i.e., it solves the following dynamic program:

$$\max_{\{I_t\}_0^\infty} E_0 \left[\int_0^\infty e^{-rt} CF_t dt - \int_0^\infty e^{-rt} P_t dI_t \right] \tag{5}$$

$$s.t.\ dK_t = -\delta K_t dt + \sigma_K K_t dz_K + dI_t,$$

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X dz_X,$$

$$\frac{dP_t}{P_t} = \mu_P dt + \sigma_P dz_P,$$

$$I_{0-} = 0,\ dI_t \geq 0\ \forall t \geq 0,$$

where r is the firm's discount rate assumed yo be constant.[17]

Let $V_t(X_t, P_t, K_t)$ denote the firm's value at time t assuming that it has followed the optimal investment policy, $I_t^* (X_t, P_t, K_t)$, in the past. To summarize, the model discussed in this section generalizes the one in Dixit and Pindyck (1994, Chapter 11) to allow for stochastic shocks to the productivity of assets (dz_K) and to the price of new assets (dz_P). Accordingly, the state variable in our model is three dimensional - (X_t, P_t, K_t). While the firm makes investment decisions based on all information available up to the current date, we assume that the information set of investors who are valuing the firm is imperfect. Specifically, while the investors know the main parameters of the firm's economic environment, such as the elasticity of cash flow to capital (α) and the drift, variance and correlation parameters of all stochastic processes (μ_X, σ_X^2, μ_P, σ_P^2, δ, σ_K^2, ρ_{XP}, ρ_{KP}, ρ_{XK}), they do not directly observe the current state variable (X_t, P_t, K_t). To estimate the value of the firm, they must rely on the information in the firm's financial statements. In the next section, we discuss the information available to investors under alternative accounting rules.

3 Investors' Information Set

The information set of investors at time t includes the current and all past financial statements of the firm.[18] Formally, we write the investors' information set as:

[17] Without loss of generality, we assume that the firm starts it operations at date 0 without any assets in place, i.e., $I_{0-} = 0$.

[18] In contrast, Nezlobin (2012) assumed that investors observe only the current financial statements and do not have access to the firm's investment history.

$$\mathcal{I}_t = \left[\{B_\tau\}_{\tau\leq t}, \{CF_\tau\}_{\tau\leq t}, \{P_\tau \cdot dI_\tau\}_{\tau\leq t}\right].$$

Here, B_τ denotes the book value of assets at time τ, CF_τ is the firm's operating cash flow at time τ, and $P_\tau \cdot dI_\tau$ is the investment cash outflow.[19] It is useful to note that the information in past income statements is subsumed by $\mathcal{I}_t$: the firm's net income from time $\tau - dt$ to τ is given by:

$$CF_\tau dt - P_\tau dI_\tau - B_{\tau-dt} + B_\tau.$$

Since $\mathcal{I}_t$ includes the whole history of book values and operating and investment cash flows, the path of the firm's net income is also in $\mathcal{I}_t$.

We now turn to characterizing the book value of assets under alternative accounting rules. Under *cash accounting*, the book value of assets is always set to zero, $BV_\tau^{cash} = 0$; i.e., the investors only observe the firm's cash flows. In our model, the net cash flows, $CF_t - P_t dI_t$, are disbursed to shareholders immediately, so the firm's cash balance is zero at all times.

Next consider *replacement cost* accounting where

$$B_t^{rc} \equiv P_t \cdot K_t$$

for all t. Under this rule, at the time of acquisition, new assets are recorded at their cost, $P_t \cdot dI_t$.[20] At each date subsequent to initial recognition, the book value of assets reflects past shocks to both the productivity of assets in place (dz_K) and to the price of new assets (dz_P). In particular, if the price of new capital goods has changed since the time of last investment, all of the firm's existing assets must be revalued accordingly. The book value of assets in place under replacement cost accounting reflects the amount that the firm would have to pay today for new capital goods to replicate the current capacity of its assets purchased in the past. This definition is consistent with the cost approach to fair value measurement as

[19] The amount of gross investment at time t is given by

$$GI_t \equiv \int_0^t P_\tau \cdot dI_\tau.$$

Since the whole history of investments, $\{P_\tau \cdot dI_\tau\}_{\tau\leq t}$, is in $\mathcal{I}_t$, investors observe the full path of the gross investment function.

[20] It will be shown that when the firm invests, the NPV of its investment is strictly positive (otherwise, it would be optimal to postpone the investment). Under replacement cost accounting, assets are capitalized at their acquisition cost, which is less than the present value of cash flows that they are expected to generate.

defined in IFRS 13.

It is useful to note that the information set of investors is imperfect even if replacement cost accounting is used to calculate the book value of assets. To see this, assume that the firm has not been investing for some time. Then, investors effectively observe the evolution of the firm's operating cash flows, $CF_t = X_t \cdot K_t^{\alpha}$, and the book value of assets, $B_t^{rc} = P_t \cdot K_t$. Even though the complete paths of these two processes are revealed to investors over time, it is still impossible for them to solve for the underlying state variable, (X_t, P_t, K_t), because it has more dimensions then their information set.[21]

Another accounting system considered in this paper is *value in use* accounting where the book value of assets at each point in time, B_t^{viu}, is set equal to the present value of cash flows that these assets are expected to generate in the future. To calculate the present value of cash flow attributable to the firm's existing assets at time t, we assume that $I_{t+\tau} = 0$ for $\tau > 0$, i.e., the firm will not make additional investments after date t. The following observation provides an expression for value in use.

Observation 1. *The present value of cash flows from the firm's assets in place at time t is given by*

$$B_t^{viu} = \frac{X_t K_t^{\alpha}}{\bar{r}}, \tag{6}$$

where

$$\bar{r} \equiv r - \mu_X + \alpha\delta - \alpha\rho_{KX}\sigma_K\sigma_X - \frac{\alpha(\alpha-1)}{2}\sigma_K^2. \tag{7}$$

The present value of cash flows from existing assets can be calculated by capitalizing the current operating cash flow, $X_t K_t^{\alpha}$, with an adjusted interest rate $\bar{r}$. This rate has an intuitive interpretation. Assume first that there is no physical depreciation of capital goods: $\delta = \sigma_K^2 = 0$. Then, the rate $\bar{r}$ reduces to the one corresponding to the Gordon growth model, $\bar{r} = r - \mu_X$. The third term in (7) reflects the effect of the expected physical depreciation of assets (δ) taking into account the concavity of the firm's cash flows in the capital stock (α). When depreciation is stochastic ($\sigma_K^2 > 0$), the concavity of cash flows in the capital stock leads to a Jensen's inequality effect: the expected value of future cash flows declines in σ_K^2.

[21] In contrast, in the model of Dixit and Pindyck (1994, Chapter 11), observing the firm's current operating cash flow and the history of investments would be sufficient to solve for the state variable.

Accordingly, the last term in (7) serves as the concavity adjustment to $\bar{r}$.[22] Lastly, a positive correlation between shocks to X_t and K_t has a positive effect on the expected value of $X_t K_t^\alpha$ (holding the mean and variance of shocks fixed); this effect is captured by the penultimate term in (7). Note that if depreciation is non-stochastic ($\sigma_K = 0$), then the last two terms in (7) are zero, and uncertainty about future demand does not affect the capitalization factor in the calculation of value in use.

The concept of value in use will prove instrumental in interpreting the expression for the firm's value that we derive below. However, it is clear from equation (6) that financial statements prepared under value in use accounting do not provide information useful to investors over and above the firm's operating cash flow. Since we assume that investors know all parameters of the firm's economic environment (except for the realizations of stochastic processes), they can calculate $\bar{r}$ and infer B_t^{viu} as

$$B_t^{viu} = \frac{CF_t}{\bar{r}}.$$

Therefore the bounds that investors can derive on the firm's equity value will be precisely the same under value in use accounting as under cash accounting.

We model *historical cost* accounting as a system under which assets are capitalized at cost when they are acquired, and, at each date thereafter, their book value reflects their current productive capacity. Therefore, we assume that shocks to asset productivity are timely reflected in the book value of assets, while the shocks to the price of new assets are not.[23] Formally, the book value of assets under historical cost accounting evolves according to the following process:

[22] Recall that $0 < \alpha < 1$, so $\frac{\alpha(\alpha-1)}{2} < 0$. Therefore, $\bar{r}$ increases in σ_K^2, and the present value of cash flows from assets in place decreases in σ_K^2.

[23] This definition can be consistent with fixed depreciation schedules only if the physical depreciation of assets is non-stochastic ($\sigma_K = 0$). Since in our model investors observe all cash flows and all parameters of the economic environment of the firm, historical cost accounting in conjunction with fixed depreciation schedules will not generate information incrementally useful to investors if $\sigma_K > 0$. In fact, we show below that *even* if shocks to asset productivity are immediately incorporated in the book value of assets, the range of valuations consistent with the firm's fundamentals is the same under historical cost accounting as under cash accounting.

$$
\begin{aligned}
dB_t^{hc} &= -\delta B_t^{hc} dt + \sigma_K B_t^{hc} dz_K + P_t dI_t, \\
B_{0-}^{hc} &= 0.
\end{aligned}
$$

The last accounting system that we consider in this paper is a variant of replacement cost accounting combined with asymmetric recognition of gains and losses. Specifically assume that all capital goods are initially recognized at their acquisition cost, $P_t dI_t$. Then, at each date, the firm compares the current carrying value of its assets to their total replacement cost: if the former amount exceeds the latter, then all assets are written down to their current replacement cost. However, if the opposite relation holds, then no write-up is recognized. Under such system, the amount of accumulated depreciation (and revaluations) is strictly increasing over time and, at time t, is equal to:

$$\sup_{\tau \leq t} \left(GI_\tau - P_\tau X_\tau \right)^+ ,$$

where GI_τ is the gross investment up to time τ and $(x)^+ = \max\{x, 0\}$. Therefore, under this rule, the book value of assets at time t, B_t^{cc}, is given by:

$$B_t^{cc} = GI_t - \sup_{\tau \leq t} \left(GI_\tau - P_\tau X_\tau \right)^+ . \tag{8}$$

We will refer to the system described above as *conditionally conservative* accounting. For our future discussion, it is important to observe that

$$B_t^{cc} \leq P_t K_t \tag{9}$$

for all t, and $dB_t^{cc} < 0$ only when $B_t^{cc} = P_t K_t$.[24]

[24] To verify inequality (9), substitute the definition of B_t^{cc} from equation (8) into the left-hand side:

$$GI_t - \sup_{\tau \leq t} \left(GI_\tau - P_\tau X_\tau \right)^+ \leq P_t K_t,$$

or, equivalently,

$$GI_t - P_t K_t \leq \sup_{\tau \leq t} \left(GI_\tau - P_\tau X_\tau \right)^+ .$$

The inequality above must hold by the definition of supremum. To verify the second statement, note that GI_t is monotonically increasing, therefore $dB_t^{cc} < 0$ implies that the supremum in the right-hand side of (8)

4 The Firm's Value

We now turn to characterizing the firm's optimal investment policy and the firm's equity value. Let ω_t denote the ratio of the firm's operating cash flow to the replacement cost of its assets in place:

$$\omega_t \equiv \frac{CF_t}{P_t K_t} = \frac{X_t K_t^{\alpha-1}}{P_t}.$$

We will refer to ω_t as the cash return on (economic) assets. Note that when $dI_t = 0$ (the firm is not investing), the three processes, X_t, K_t, and P_t, evolve as geometric Brownian motions, and therefore so does ω_t. Cash return on assets increases in demand for the firm's output, X_t, and decreases in the price of new capital goods, P_t. Furthermore, since $\alpha < 1$, ω_t decreases in the firm's capital stock. In particular, the firm's investment, $dI_t > 0$, has a negative effect on the cash return on assets ratio due to diminishing returns to capital inherent in the firm's production function.

We show in the proof of Proposition 1 that the firm's optimal investment policy is fully characterized by a certain threshold value of the cash return on assets ratio, ω^*, that serves as a reflecting barrier for the process ω_t. When cash return on assets is below this barrier, $\omega_t < \omega^*$, the firm does not invest. When cash return on assets reaches the barrier ω^*, the firm makes a sequence of investments that is just sufficient to prevent ω_t from crossing the threshold; the process ω_t is thus reflected from the barrier as illustrated in Figure 1 below. Intuitively, the firm expands when the product market is sufficiently profitable; as the firm increases its capacity, the marginal, as well as average, profitability of its sales falls and the firm returns to the inaction (no-investment) region. After that point, new investments will be made only when the output market expands sufficiently more to push ω_t to the barrier again. Therefore, after the initial investment at date 0, the cash return on assets ratio will follow a geometric Brownian motion, reflected at ω^*.

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is attained at t:

$$\sup_{\tau \le t} (GI_\tau - P_\tau X_\tau)^+ = GI_t - P_t X_t.$$

The equation above is equivalent to $B_t^{cc} = P_t K_t$.

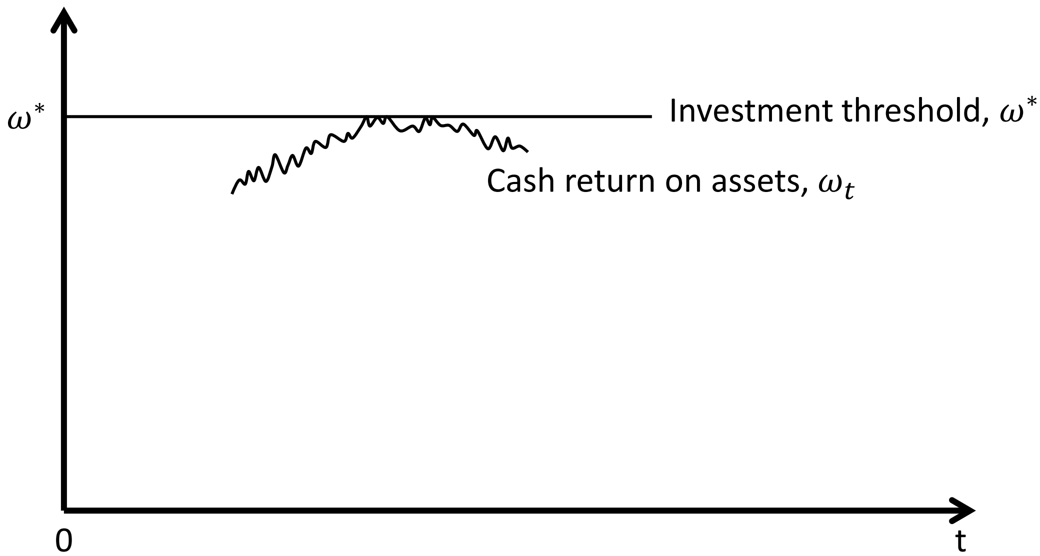

Figure 1: **Optimal Investment Policy**

We will now write down the Hamilton-Jacobi-Bellman equation that determines the firm's value function, $V(X, P, K)$.[25] First, assume that the firm is not investing, $dI_t = 0$. Then, by Ito's lemma, we can write $E[CFdt + dV]$ (i.e., the instantaneous dividend plus the expected change in the value of future cash flows) in the inaction region as:

$$\mathcal{L}V \cdot dt,$$

where

$$\mathcal{L}V \equiv XK^{\alpha} - \delta K V_K + \mu_X X V_X + \mu_P P V_P + \frac{1}{2}\sigma_K^2 K^2 V_{KK} + \frac{1}{2}\sigma_P^2 P^2 V_{PP} + \frac{1}{2}\sigma_X^2 X^2 V_{XX} +$$

$$\rho_{KP}\sigma_K\sigma_P KPV_{KP} + \rho_{KX}\sigma_K\sigma_X KXV_{KX} + \rho_{XP}\sigma_X\sigma_P PXV_{XP}.$$

On the other hand, since V is the present value of expected cash flows, it has to satisfy:

$$E[CFdt + dV] = rV \cdot dt.$$

[25] To simplify notation, we will drop the subscript t when it is not needed, and use subscripts on V to denote partial derivatives, e.g., V_{PX} in the expression below denotes the cross-derivative of V with respect to P and X.

Therefore, in the inaction region, we have:

$$rV = \mathcal{L}V.$$

If the firm's investment is positive, $dI_t > 0$, then $rV > \mathcal{L}V$, since the right-hand side does not include the (positive) NPV of the marginal investment project.

At times when the firm invests, the amount of investment is chosen so that the unit price of new capital goods is equal to the marginal benefit of investment,

$$P = V_K.$$

The firm is in the inaction region when the cost of new capital goods is greater than the marginal increase in value that a new investment would generate, $P - V_K > 0$. To summarize the observations above, the firm's value function is determined by the following variational inequality:

$$\min \left(rV - \mathcal{L}V, P - V_K\right) = 0 \tag{10}$$

In the investment region, the first term is positive and the second term is equal to zero; in the inaction region, $rV - \mathcal{L}V = 0 < P - V_K$.

Our first proposition provides an expression for the firm's equity value at each date. In formulating this proposition, the following notation will be convenient. First, let $\bar{\mu}$ be equal to:

$$\begin{aligned}\bar{\mu} \equiv\ & \mu_X - \mu_P - (\alpha - 1)\,\delta + \frac{1}{2}\sigma_X^2 + \frac{1}{2}\sigma_P^2 + (\alpha - 1)\left(\alpha - \frac{1}{2}\right)\sigma_K^2 + \\ & (\alpha - 1)\,\rho_{KX}\sigma_K\sigma_X - \rho_{PX}\sigma_P\sigma_X - \alpha\rho_{KP}\sigma_K\sigma_P.\end{aligned} \tag{11}$$

Note that in the absence of uncertainty, $\bar{\mu}$ would be equal to $\mu_X - \mu_P - (\alpha - 1)\,\delta$, i.e., it would be equal to the growth rate of the cash return on assets ratio, ω. Next, let $\bar{\sigma}^2$ be the following measure of aggregate uncertainty of the firm's economic environment:

$$\bar{\sigma}^2 \equiv \sigma_X^2 + \sigma_P^2 + (\alpha - 1)^2\,\sigma_K^2 + 2\,(\alpha - 1)\,\rho_{KX}\sigma_K\sigma_X - 2\rho_{PX}\sigma_P\sigma_X - 2\,(\alpha - 1)\,\rho_{PK}\sigma_P\sigma_K. \tag{12}$$

Lastly, let λ and A be

$$\lambda \equiv -\frac{\bar{\mu}}{\bar{\sigma}^2} + \sqrt{\left(\frac{\bar{\mu}}{\bar{\sigma}^2}\right)^2 + \frac{2\bar{r}}{\bar{\sigma}^2}} > 0,$$

and

$$A \equiv \frac{\alpha}{(\lambda+1)(\lambda - \alpha\lambda - \alpha)},$$

where $\bar{r}$ is given by equation (7). Note that $\bar{\mu}$, $\bar{\sigma}^2$, λ, A depend only on the parameters of the model that are known to investors and do not depend on the realizations of stochastic processes, $\{X_t, P_t, K_t\}$.[26]

Proposition 1. *The firm's equity value at time t is equal to*

$$V_t = \frac{CF_t}{\bar{r}}\left(1 + A \cdot \left[\frac{CF_t}{B_t^{rc} \cdot \omega^*}\right]^{\lambda}\right), \tag{13}$$

where ω^* is the optimal threshold for investment, given by:

$$\omega^* = \frac{\bar{r} \cdot (1+\lambda)}{\alpha \cdot \lambda}. \tag{14}$$

The firm makes its first investment so that

$$I_0^* = \left(\frac{X_0}{P_0}\omega^*\right)^{\frac{1}{1-\alpha}},$$

and then invests only when its cash return on assets ratio, ω_t, is equal to ω^*.

The equity valuation formula in (13) has an intuitive interpretation. Recall that the expected value of cash flows from assets in place (value-in-use) is given by:

$$B^{viu} = \frac{CF_t}{\bar{r}}.$$

[26] To ensure that a solution for the firm's equity value exists and is always finite, we impose the following two constraints on the parameters of the model:

$$\lambda > \frac{\alpha}{1-\alpha}$$

and

$$\bar{r} > 0.$$

Proposition 1 shows that the firm's equity value exceeds the value of cash flows from assets in place by the expectation of payoffs from capacity expansion options. The value of these growth options is given by:

$$\frac{CF_t}{\bar{r}} \cdot A \cdot \left[\frac{CF_t}{B_t^{rc} \cdot \omega^*}\right]^{\lambda}.$$

The quantity above is proportional to the present value of cash flows from assets in place times the cash return on assets ratio raised to power λ. Growth options become more valuable as cash return on assets approaches the optimal exercise threshold, ω^*.[27] Note that term in square brackets in the expression above is equal to $\frac{\omega_t}{\omega^*}$ and thus measure how far the current cash return on assets ratio is from the investment threshold. Given replacement cost accounting, investors can precisely value both the cash flows from assets in place and the capacity expansion options.

Recall that the price of new capital goods, P_t, as well as the parameters of the $\{P_t\}$ process do not enter the expression for the present value of cash flows from existing assets. The parameters of the $\{P_t\}$ process do, however, affect the constants A, λ, and ω^*, and therefore the valuation of capacity expansion options. Importantly, the value of capacity expansion options also depends on P_t: the higher the price of new capital goods today, the less likely it is that the firm will exercise its growth options soon. Therefore the total value of the firm's equity depends on all three components of the state variable, $\{X_t, P_t, K_t\}$. Proposition 1, however, shows that investors do not need to observe the individual components of the state variable to be able to value the firm's cash flows as long as they observe the two aggregate variables, CF_t and B_t^{rc}.

While we do not explicitly model agency problems in this paper, it is interesting to note that given replacement cost accounting, investors can always *verify* that the firm is indeed following the optimal investment policy: positive investment should happen if and only if

$$\frac{CF_t}{B_t^{rc}} = \omega^*,$$

and investments should be chosen such that ω^* becomes a reflecting barrier for the cash return on investment process. Specifically, on the optimal investment path, the cash return on assets ratio must never exceeds ω^*, and the firm must not invest when $\omega < \omega^*$. This finding

[27] It is straightforward to verify that when the price of new assets is constant over time and the physical depreciation is non-stochastic, the expression for the optimal investment barrier in (14) reduces to the one in Dixit and Pindyck (1994, p. 376).

is consistent with the recent results in the managerial performance evaluation literature that show that replacement cost accounting can be used to achieve goal congruence in settings with overlapping investments; see, for instance, Rogerson (2008) and Nezlobin et al. (2015).

We now turn to characterizing the bounds on the firm equity value that investors can calculate under accounting systems that do not convey enough information to value the firm precisely. First, note that immediately after (or very soon after) observing a positive investment, the cash return on economic assets is close to ω^*. Therefore, even without directly observing the replacement cost of assets in place, investors know that when $P_t dI_t > 0$,[28]

$$V_t = \frac{CF_t}{\bar{r}} (1 + A) . \tag{15}$$

On the other hand, if $P_t dI_t = 0$, it has to be that $\omega_t < \omega^*$. The firm's value must then be bounded from below by the present value of expected cash flows from assets in place, and from above by the value of the firm if it were at the investment threshold:

$$\frac{CF_t}{\bar{r}} \leq V_t \leq \frac{CF_t}{\bar{r}} (1 + A) . \tag{16}$$

It is straightforward to see that if investors only observe the firm's cash flows, then the bounds in (16) are the tightest possible based on the investors' information set, $\mathcal{I}_t$. Indeed, if P_t is very large, then the value of growth options approaches zero and $V_t \to \frac{CF_t}{\bar{r}}$. On the other hand, the price of new assets can be arbitrarily close to

$$\frac{CF_t}{K_t \cdot \omega^*},$$

in which case V_t is going to be very close to $\frac{CF_t}{\bar{r}} (1 + A)$. Given cash accounting, the changes in P_t since the firm's last investment do not get reflected in $\mathcal{I}_t$. Therefore, any inequality on V_t that is tighter than (16) will be violated with positive probability.

Since, according to Observation 1, the present value of cash flows from the firm's assets in place is collinear with CF_t, the bounds in (16) also cannot be improved under value in use accounting. Similarly, if the firm uses historical cost accounting, then changes in P_t when $\omega_t < \omega^*$ do not affect any of the observable variables in $\mathcal{I}_t$. Therefore we obtain the following corollary.

[28] Recall that investors observe $P_t \cdot dI_t$ as this is the investment cash outflow.

Corollary 1. *Given historical cost accounting, value in use accounting, or cash accounting, the tightest bounds on the firm's equity value that hold almost surely conditional on $\mathcal{I}_t$ are:*
If $P_t dI_t > 0$,

$$V_t = \frac{CF_t}{\bar{r}} \left(1 + A\right);$$

if $P_t dI_t = 0$,

$$\frac{CF_t}{\bar{r}} \leq V_t \leq \frac{CF_t}{\bar{r}} \left(1 + A\right).$$

Given conditionally conservative accounting, investors know that the book value of assets understates their replacement cost, and therefore

$$\omega_t \leq \frac{CF_t}{B_t^{cc}}.$$

Therefore, the following upper bound on the firm's value must always hold:

$$V_t \leq \frac{CF_t}{\bar{r}} \left(1 + A \cdot \left[\frac{CF_t}{B_t^{cc} \cdot \omega^*}\right]^{\lambda}\right).$$

If the firm is in the inaction region, the lower bound on the firm's value from inequality (16) cannot be improved: as the price of new capital goods goes up, the firm's value can get arbitrarily close to $\frac{CF_t}{\bar{r}}$, and the price increases in the capital goods market do not get reflected in $\mathcal{I}_t$ under conditionally conservative accounting. However, if investors observe a write-down, $dB_t^{cc} < 0$, then it has to be that $B_t^{cc} = P_t K_t$, and the firm can be valued precisely. These observations are summarized in the following corollary.

Corollary 2. *Given conditionally conservative accounting, the tightest bounds on the firm's equity value that hold almost surely conditional on $\mathcal{I}_t$ are:*
If $P_t dI_t > 0$,

$$V_t = \frac{CF_t}{\bar{r}} \left(1 + A\right);$$

if $dB_t^{cc} < 0$,

$$V_t = \frac{CF_t}{\bar{r}} \left(1 + A \cdot \left[\frac{CF_t}{B_t^{cc} \cdot \omega^*}\right]^{\lambda}\right);$$

if $P_t dI_t = 0$ *and* $dB_t^{cc} \geq 0$,

$$\frac{CF_t}{\bar{r}} \leq V_t \leq \frac{CF_t}{\bar{r}} \left(1 + A \cdot \left[\frac{CF_t}{B_t^{cc} \cdot \omega^*}\right]^{\lambda}\right).$$

While the actual accounting standards (such as U.S. GAAP and IFRS) are generally different from the stylized accounting rules modeled here, they share certain common features. For example, under IAS 36, firms are required to recognize an impairment if the carrying amount of an asset exceeds the higher of its fair value (which, for an illiquid asset, can be measured as its replacement cost) and its value in use. In Appendix B, we provide bounds on the equity value for firms that account for their property, plant and equipment using the cost model of IAS 16 and recognize write-downs according to IAS 36.

5 Reversible Investments

In this section, we assume that the firm's investments are reversible, i.e., the firm can sell its used capital goods at a price of new capital goods with equivalent productive capacity. For technical reasons, we will now consider the optimal investment and equity valuation problems in discrete time.[29] Consistent with previous sections, we allow for stochastic physical depreciation, stochastic demand and stochastic price of new capital goods.

Let period t be the interval of time between dates $t-1$ and t. We assume that the cash flow from operations, $CF_t = X_t K_t^{\alpha}$, and the investment cash outflow, $P_t I_t$, are realized at the end of period t (i.e., just before date t). For notational convenience, we will focus on the cum-dividend value of the firm defined as:

$$V_t = CF_t - P_t I_t^* + \sum_{\tau=1}^{\infty} \beta^{\tau} \cdot E\left[CF_{t+\tau} - P_{t+\tau} I_{t+\tau}^*\right],$$

[29] When the firm can adjust its capacity both upwards and downwards, the optimal investment policy is to set the marginal revenue equal to the (properly defined) user cost of capital at all times. It can then be verified that on the optimal investment path, the controlled process, K_t, follows a geometric Brownian motion, and the control process, I_t, has an unbounded variation. However, the standard approach in the singular stochastic control theory limits the set of controls to processes of bounded variation (see Fleming and Soner, 2006, Chapter 8). Therefore, we will first solve the model in discrete time; once we obtain a solution, we will describe its limit as the duration of the time period approaches zero for the purposes of comparison.

where $I^*_{t+\tau}$ denotes the optimal investment policy and $\beta = \frac{1}{1+r}$ is the firm's discount factor.

We assume the following evolution of stochastic processes, X_t, P_t, K_t:

$$X_t = g_{X,t} \cdot X_{t-1},$$

$$P_t = g_{P,t} \cdot P_{t-1},$$

$$K_t = g_{K,t} \cdot \left[(1-\delta) \cdot K_{t-1} + I_{t-1}\right],$$

where $g_{X,t}$, $g_{P,t}$, $g_{K,t}$ are random variables realized in period t before the operating cash flow of that period is observed and the new investment I_t is chosen. Let $\theta_t \equiv (g_{X,t}, g_{P,t}, g_{K,t})$ denote the three-dimensional innovation in the state variable (X_t, P_t, K_t). We assume that θ_t are i.i.d. over time, but the components of each θ_t can be mutually correlated. Since the distribution of θ_t is time-invariant, we will drop the time subscript when we refer to expectations of new innovations, e.g., we will write $E\left[g_X\right]$ for $E_{t-1}\left[g_{X,t}\right]$ and $E\left[g_X g_K^\alpha\right]$ for $E_{t-1}\left[g_{X,t} g_{K,t}^\alpha\right]$.

Let $\hat{K}_t \equiv (1-\delta) \cdot K_{t-1} + I_{t-1}$, then we have:

$$K_t = g_{K,t} \cdot \hat{K}_t.$$

Note further that the firm's investment in period t can be expressed as:

$$I_t = \hat{K}_{t+1} - (1-\delta)\,\hat{K}_t g_{K,t}. \tag{17}$$

Intuitively, $\hat{K}_{t+1}$ is the firm's expectation (at the end of period t) of the productive capacity of its assets in period $t+1$.[30] Note that $\hat{K}_{t+1}$ includes the productive capacity of the assets just purchased, I_t. The total replacement cost of assets in place at date t can therefore be written as:

$$RC_t = P_t \cdot \hat{K}_{t+1}.$$

In formulating the Bellman equation for the present value of the firm's optimized cash flows, it will be convenient to write V_t as $V_t\left(\hat{K}_t, X_t, P_t, g_{K,t}\right)$. Then, the Bellman equation

[30] Since we allow for stochastic depreciation, the firm does not know the exact capacity of its assets in period $t+1$ before $g_{K,t+1}$ is realized.

takes the following form:

$$V_t\left(\hat{K}_t, X_t, P_t, g_{K,t}\right) = CF_t + \max_{I_t}\left\{\beta \cdot E\left[V_{t+1}\left(\hat{K}_{t+1}, X_{t+1}, P_{t+1}, g_{K,t+1}\right)\right] - P_t \cdot I_t\right\}.$$

Now applying (17), we can simplify the equation above as:

$$V_t\left(\hat{K}_t, X_t, P_t, g_{K,t}\right) = X_t g_{K,t}^{\alpha}\hat{K}_t^{\alpha} + P_t\left(1-\delta\right)\hat{K}_t g_{K,t} + \tag{18}$$

$$+\max_{\hat{K}_{t+1}}\left\{\beta \cdot E\left[V_{t+1}\left(\hat{K}_{t+1}, X_{t+1}, P_{t+1}, g_{K,t+1}\right)\right] - P_t \cdot \hat{K}_{t+1}\right\}.$$

To solve the equation above we will impose two regularity conditions. First, to ensure that the firm's valuation is always finite, we will assume that X_t does not grow too quickly relative to P_t:

$$1 + r > E\left[g_X^{\frac{1}{1-\alpha}} g_P^{-\frac{\alpha}{1-\alpha}}\right].$$

Second, if the price of new assets is expected to grow very fast, then the firm could make infinite profits by buying capital goods today and selling them in future periods. To avoid such behavior, we assume that

$$1 + r > \left(1-\delta\right) E\left[g_K g_P\right].$$

The following proposition provides an equity valuation formula for a firm that makes reversible investments.

Proposition 2. *The firm's cum-dividend equity value at time t is equal to*

$$V_t = CF_t - P_t I_t + \left(1 + C_1\right) \cdot RC_t, \tag{19}$$

and the firm's optimal investment policy is characterized by:

$$\hat{K}_{t+1} = C_2 \cdot \left(\frac{X_t}{P_t}\right)^{\frac{1}{1-\alpha}}, \tag{20}$$

where C_1 and C_2 are two non-negative constants that depend only on the parameters of the *stochastic processes X_t,P_t,K_t and not on their realizations.*[31]

The valuation equation in (19) states that the cum-dividend value of the firm at date t is equal to the current cash flow ($CF_t - P_t I_t$) plus another term that is proportional to the replacement cost of assets in place; in other words, the ex-dividend value of the firm equity is equal to $(1 + C_1) \cdot RC_t$. In the investment literature, it is common to decompose the value of the firm's equity into two components: the replacement cost of its assets in place and the discounted sum of future economic profits (see, e.g. Thomadakis 1976, Lindenberg and Ross 1981, Fisher and McGowan 1983, Salinger 1984, and Abel and Eberly 2011). The constant C_1 in our model reflects the ratio of the present value of future economic profits to the replacement cost of assets in place. It is important to note that C_1 does not depend on the current state variable (X_t, P_t, K_t). All the value-relevant information about the underlying state is summarized in the replacement cost of the firm's assets in place, RC_t.

It is interesting to compare our firm valuation equations in Propositions 1 and 2. It turns out that replacement cost of assets in place is an important variable for equity valuation in both scenarios.[32] However, as discussed above, the replacement cost of assets in place alone is a sufficient statistic for the firm's equity value when investments are reversible. In contrast, according to Proposition 1, to value a firm with irreversible investments in the inaction region, one needs to observe both the replacement cost of its assets and its operating cash flow.

To understand this difference between the two results, it is useful to recall that the underlying state in both scenarios is three-dimensional, i.e., it is determined by variables X_t, P_t, K_t. The process $\{K_t\}$ is controlled: it is affected by the firm's investments. When the firm can adjust its capital stock in both directions (investments are reversible), the firm ensures that the marginal expected benefit of investment is precisely equal to the marginal

[31] We show in the proof of Proposition 2 that the constants C_1 and C_2 are given by:

$$C_2 \equiv \left(\frac{\alpha \beta E\left[g_X g_K^{\alpha}\right]}{1 - \beta\left(1-\delta\right) E\left[g_K g_P\right]} \right)^{\frac{1}{1-\alpha}}$$

and

$$C_1 \equiv \frac{\left(1-\alpha\right)\left(1 - \beta\left(1-\delta\right) E\left[g_K g_P\right]\right)}{\alpha \left(1 - \beta E\left[g_X^{\frac{1}{1-\alpha}} g_P^{-\frac{\alpha}{1-\alpha}}\right]\right)}.$$

We will discuss below the limiting values of these constants as the length of the time period approaches zero.

[32] In fact, it is straightforward to verify that if investments are reversible, it is not possible to value the firm's equity precisely without observing the replacement cost of its assets.

cost of new assets, P_t, at each date. Therefore, the three processes, X_t, P_t and the expected forward capacity ($\hat{K}_{t+1}$) are always linked by equation (20). This effectively eliminates one dimension from the underlying state variable, and, relative to the scenario with irreversible investments, it is sufficient for investors to observe one financial variable less to be able to value the firm precisely.[33]

In contrast, when the firm's investments are irreversible and the firm is in the inaction region, the three components of the state variable, X_t,P_t,K_t, move independently: the firm cannot adjust its capacity downwards and the output market is not profitable enough to justify additional investments, so dI_t is zero. Then, investors need to have more information to value the firm. Accordingly, the valuation equation in Proposition 1 depends on both CF_t and P_tK_t. However, when the firm reaches its investment boundary, investors know that an additional constraint on the components of the state variable is binding, $\omega_t = \omega^*$, and the firm's equity can be valued based on just one financial variable as Corollary 1 demonstrates.

Propositions 1 and 2 also show that the replacement cost of assets in place affects the value of the firm's equity differently in the two scenarios. First, note that the value of a firm with reversible investments always (weakly) exceeds the replacement cost of its assets, $V_t \geq RC_t$. This has to be the case at all times because otherwise the firm could simply sell all of its assets today and generate more value than by participating in the output market in future periods. The same inequality, however, does not necessarily hold for a firm with irreversible investments. In particular, note that the expression for V_t in Proposition 1 approaches zero as $X_t \to 0$; so for small values of X_t, the firm's value will be less than the replacement cost of its assets.

Second, Proposition 1 states that, controlling for the current operating cash flow, the value of the firm with irreversible investments decreases in RC_t. In contrast, in the scenario with reversible investments, V_t strictly increases in RC_t. To understand this difference, consider what happens after a positive shock to the price of new assets, P_t. In both cases, the value of the firm's future cash flows goes down, since the cost of its inputs went up. In the scenario with irreversible investments, the replacement cost of the firm's assets, P_tK_t, goes up since the firm cannot change K_t in response to the change in P_t. Therefore, an

[33] However, observing cash flows alone is not sufficient for equity valuation purposes in the scenario with reversible investments. According to equation (19), the firm's equity value depends on $P_t\hat{K}_{t+1}$, but investors cannot solve for this quantity if they observe only $X_tK_t^{\alpha}$ and P_tI_t. Formally, one can verify that there can exist two firms with the same histories of cash flows up to date t but different replacement costs of assets at date t and, consequently, with different valuations at that date.

increase in P_t (an event unfavorable to the firm) has an effect of *increasing* the replacement cost of its assets in place, RC_t. In the scenario with reversible investments, the firm can sell some of its assets at the new price, P_t, and choose a new level of capital stock. In fact, according to equation (20), the firm's replacement cost of assets at date t is:

$$P_t \hat{K}_{t+1} = C_2 \cdot X_r^{\frac{1}{1-\alpha}} P_t^{-\frac{\alpha}{1-\alpha}}.$$

The quantity above decreases in P_t. Therefore, in the scenario with reversible investments, an increase in the price of new capital goods (which is, again, an event unfavorable to the firm) leads to a *decrease* in the replacement cost of assets after taking into account the firm's optimal capacity adjustment. To summarize, our results indicate that while the replacement cost of assets in place is an important variable for equity valuation in both scenarios, the exact way this variable enters the equity value function crucially depends on the reversibility of investments.

In 1980's, under SFAS No. 33, large firms were required to disclose the current (replacement) cost of their assets such as plant, property, and equipment and inventories.[34] Several papers failed to find incremental value of SFAS 33 disclosures over the historical cost earnings (see, e.g., Beaver and Landsman 1983 and Beaver and Ryan 1985). However, Revsine (1973) suggested that positive holding gains under replacement cost accounting can be good news for some firms and bad news for others, depending on how well each firm is able to react to price changes in its input markets. Hopwood and Schaefer (1989) provide empirical support for Revsine's argument. Consistent with the intuition described in this literature, our model formally shows that the direction of the relation between equity value and replacement cost of assets depends critically on the firm's ability to adjust its capital stock downwards. Our results also help explain the small or negative coefficients on the book value of assets in equity valuation regressions; see, for example, Dechow et al. (1999) and Hao et al. (2011).

To conclude this section, we describe the limits of the coefficients C_1 and C_2 as the length of the time period approaches zero. It can be verified that as $dt \to 0$, the firm's value

[34] These disclosures were used by, e.g., Lindenberg and Ross (1981) to construct their Tobin's q estimates.

becomes:[35]

$$V_t = \left(1 + \hat{C}_1\right) \cdot RC_t$$

where

$$\hat{C}_1 = \frac{(1-\alpha)}{\alpha} \cdot \frac{r + \delta - \mu_P - \frac{1}{2}\sigma_K^2 - \rho_{PK}\sigma_P\sigma_K}{r - \frac{(\mu_X - \alpha\mu_P)}{1-\alpha} - \frac{\alpha}{2(1-\alpha)^2}\left(\sigma_X^2 + \sigma_P^2 - 2\rho_{XP}\sigma_X\sigma_P\right)}, \tag{21}$$

and the firm's optimal investment policy is such that

$$K_t = \hat{C}_2 \cdot \left(\frac{X_t}{P_t}\right)^{\frac{1}{1-\alpha}},$$

where

$$\hat{C}_2 = \left(\frac{\alpha}{r + \delta - \mu_P - \frac{1}{2}\sigma_K^2 - \rho_{PK}\sigma_P\sigma_K}\right)^{\frac{1}{1-\alpha}}. \tag{22}$$

In the expressions above, the constants $\mu_X, \mu_P, \sigma_X, \sigma_P, \sigma_K, \rho_{PK}, \rho_{XP}$ are defined consistently with the specification in Section 2.

The expressions in (21) and (22) can be intuitively interpreted in certain special cases. Assume, for example, that depreciation is non-stochastic, $\sigma_K = 0$, and there is no drift in the price of new assets, $\mu_P = 0$. Then, $\hat{C}_2$ becomes:

$$\hat{C}_2 = \left(\frac{\alpha}{r + \delta}\right)^{\frac{1}{1-\alpha}}.$$

The denominator of the ratio in brackets, $r + \delta$, is the standard user cost of capital (see, e.g., Jorgensen 1963 and Abel and Eberly 2011). The remaining terms in the denominator of $\hat{C}_2$ in (22) adjust the user cost of capital for uncertainty in the prices of new capital goods and the productive capacity of assets in place.

Now assume that there is no uncertainty about future values of X_t, P_t, K_t and $\mu_P = \mu_X = 0$, and consider the expression for the firm's equity value:

$$V_t = \left(1 + \frac{(1-\alpha)(r+\delta)}{\alpha r}\right) P_t K_t.$$

Intuitively, the firm's equity value is equal to the replacement cost of its assets, $P_t K_t$, plus

[35] When $dt \to 0$, there is no difference between the cum-dividend and ex-dividend valuations of the firm since the instantaneous cash flows approach zero. The proof of expressions (21) and (22) is available from authors upon request.

the present value of future economic profits:

$$\frac{(1-\alpha)(r+\delta)}{\alpha r} P_t K_t.$$

To see that the expression above is indeed equal to the present value of future economic profits, recall that $(r+\delta) P_t K_t$ is the user cost of capital employed, $\frac{1-\alpha}{\alpha}$ is the optimal mark-up in the firm's output market, and $\frac{1}{r}$ is the capitalization factor for a stream of payments equal in expectation. In the special case considered here, there is no drift in the price of new assets or demand for the firm's output, $\mu_P = \mu_X = 0$, so the stream of future economic profits effectively becomes an annuity.

6 Conclusion

Our paper studies the problem of equity valuation based on accounting information in a setting where the firm makes investments in capital goods, facing uncertainty regarding future conditions in its output- and capital goods markets. We initially assume that the firm's investments are irreversible and the productivity of its assets is stochastic. Our main result shows that if the firm's financial statements are prepared using replacement cost accounting, then outside investors will have sufficient information to value the firm's equity. The demand for replacement cost disclosures comes from investors' need to value the firm's growth options. In a setting with reversible investments, we show that replacement cost accounting also provides information useful for equity valuation. However, the relation between the firm's equity value and the replacement cost of its assets depends on the firm's ability to sell its used capital goods.

Our results in this paper help understand the informational needs of equity investors. However, in setting accounting standards, the regulators are concerned with interests of a broader group of financial statement users, including the firm's lenders and other creditors. Characterizing the disclosure preferences of broader groups of financial statement users in a setting with real options is an interesting direction for future research.

Appendix A

Proof of Observation 1

The present value of cash flows from existing assets is equal to:

$$B_t^{viu} = E_t\left[\int_t^{\infty} e^{-r(s-t)} X_s K_s^{\alpha} ds\right] \tag{23}$$

with the state variables following

$$dK_t = -\delta K_t dt + \sigma_K K_t dz_K,$$

$$dX_t = \mu_X X_t dt + \sigma_X X_t dz_X.$$

From the above SDE's, we can solve for X_t and K_t^{α}:[36]

$$K_t = K_0 e^{-\left(\delta + \frac{1}{2}\sigma_K^2\right)t + \sigma_K z_{K,t}},$$
$$X_t = X_0 e^{\left(\mu_X - \frac{1}{2}\sigma_X^2\right)t + \sigma_X z_{X,t}}.$$

Then, the ratio of $X_s K_s^{\alpha} / X_t K_t^{\alpha}$ is

$$\frac{X_s K_s^{\alpha}}{X_t K_t^{\alpha}} = \exp\left(-\left(-\mu_X + \alpha\delta + \frac{\alpha}{2}\sigma_K^2 + \frac{1}{2}\sigma_X^2\right)(s-t) + \alpha\sigma_K\left(z_{K,s} - z_{K,t}\right) + \sigma_X\left(z_{X,s} - z_{X,t}\right)\right). \tag{24}$$

Consider the following process:

$$\widetilde{z}_t \equiv \alpha\sigma_K z_{K,t} + \sigma_X z_{X,t} \sim \mathcal{N}\left(0, \left(\alpha^2\sigma_K^2 + \sigma_X^2 + 2\alpha\rho_{XK}\sigma_K\sigma_X\right)t\right),$$

and rewrite equation (24) as

$$\frac{X_s K_s^{\alpha}}{X_t K_t^{\alpha}} = \exp\left(-\left(-\mu_X + \alpha\delta + \frac{\alpha}{2}\sigma_K^2 + \frac{1}{2}\sigma_X^2\right)(s-t) + \widetilde{z}_s - \widetilde{z}_t\right).$$

[36] We will write $z_{K,t}$ to denote the value of process z_K at time t.

Then, the value of cash flows from assets in place can be expressed as:

$$B_t^{viu} = E_t\left[\int_t^\infty e^{-r(s-t)} X_s K_s^\alpha ds\right] = X_t K_t^\alpha \int_t^\infty e^{-r(s-t)} E_t\left[\frac{X_s K_s^\alpha}{X_t K_t^\alpha}\right] ds =$$
$$= X_t K_t^\alpha \int_t^\infty e^{-\left(r-\mu_X+\alpha\delta+\frac{\alpha}{2}\sigma_K^2+\frac{1}{2}\sigma_X^2\right)(s-t)} E_t\left[e^{\widetilde{z}_s-\widetilde{z}_t}\right] ds.$$

Since $\widetilde{z}_s - \widetilde{z}_t$ is normally distributed,

$$E_t\left[e^{\widetilde{z}_s-\widetilde{z}_t}\right] = e^{\frac{1}{2}\left(\alpha^2\sigma_K^2+\sigma_X^2+2\alpha\rho_{XK}\sigma_K\sigma_X\right)(s-t)}.$$

Therefore,

$$B_t^{viu} = X_t K_t^\alpha \int_t^\infty e^{-\left(r-\mu_X+\alpha\delta+\frac{\alpha}{2}\sigma_K^2+\frac{1}{2}\sigma_X^2\right)(s-t)} E_t\left[e^{\widetilde{z}_s-\widetilde{z}_t}\right] ds =$$
$$= X_t K_t^\alpha \int_t^\infty e^{-\bar{r}(s-t)} ds = \frac{XK^\alpha}{\bar{r}}.$$

$\square$

Proof of Proposition 1

Recall that the Hamilton-Jacobi-Bellman equation for the firm's value is:

$$\min\left(rV - \mathcal{L}V, P - V_K\right) = 0, \tag{25}$$

where

$$\mathcal{L}V \equiv XK^\alpha - \delta K V_K + \mu_X X V_X + \mu_P P V_P + \frac{1}{2}\sigma_K^2 K^2 V_{KK} + \frac{1}{2}\sigma_P^2 P^2 V_{PP} + \frac{1}{2}\sigma_X^2 X^2 V_{XX} +$$

$$\rho_{KP}\sigma_K\sigma_P KPV_{KP} + \rho_{KX}\sigma_K\sigma_X KXV_{KX} + \rho_{XP}\sigma_X\sigma_P PXV_{XP}.$$

We will first construct a solution to the variational inequality (25) that is continuously differentiable everywhere and is C^2 in the inaction region (note that that the PDE in the investment region, the second part of the variational inequality in 25, does not depend on second derivatives of V). The boundary between investment and inaction regions will be determined jointly with the solution. It will then follow from a standard verification argument that the value function so constructed is indeed equal to the present value of optimized cash flows, and the investment policy characterized by the boundary is indeed optimal.

We guess the solution of the variational inequality (25) to be of the following form,

$$V(X,P,K) = XK^{\alpha} f(X,P,K), \tag{26}$$

where $f(X,P,K)$ is to be determined. Substituting (26) into (25) results in the following variational inequality for $f(X,P,K)$

$$\min\left(\overline{r}f - \mathcal{L}^f f, \frac{P}{XK^{\alpha-1}} - Kf_K - \alpha f\right) = 0, \tag{27}$$

where

$$\mathcal{L}^f f = 1 + \bar{\mu}_K K f_K + \bar{\mu}_X X f_X + \bar{\mu}_P P f_P + \frac{1}{2}\sigma_K^2 K^2 f_{KK} + \frac{1}{2}\sigma_P^2 P^2 f_{PP} +$$

$$\frac{1}{2}\sigma_X^2 X^2 f_{XX} + \rho_{KP}\sigma_K\sigma_P KP f_{KP} + \rho_{KX}\sigma_K\sigma_X KX f_{KX} + \rho_{XP}\sigma_X\sigma_P PX f_{XP}.$$

The new coefficients $\bar{\mu}_P$, $\bar{\mu}_K$, $\bar{\mu}_X$ are defined as

$$\begin{aligned}
\bar{\mu}_P &= \mu_P + \alpha\rho_{KP}\sigma_K\sigma_P + \rho_{XP}\sigma_X\sigma_P, \\
\bar{\mu}_K &= -\delta + \alpha\sigma_K^2 + \rho_{KX}\sigma_X\sigma_K, \\
\bar{\mu}_X &= \mu_X + \alpha\rho_{KX}\sigma_K\sigma_X + \sigma_X^2.
\end{aligned} \tag{28}$$

First, we will solve the part of the variational inequality that is responsible for the no-investment region ($\overline{r}f - \mathcal{L}^f f = 0$). In the no-investment region, we have:

$$\overline{r}f = 1 + \bar{\mu}_K K f_K + \bar{\mu}_X X f_X + \bar{\mu}_P P f_P + \frac{1}{2}\sigma_K^2 K^2 f_{KK} + \frac{1}{2}\sigma_P^2 P^2 f_{PP} + \tag{29}$$

$$\frac{1}{2}\sigma_X^2 X^2 f_{XX} + \rho_{KP}\sigma_K\sigma_P KP f_{KP} + \rho_{KX}\sigma_K\sigma_X KX f_{KX} + \rho_{XP}\sigma_X\sigma_P PX f_{XP}.$$

We look for a candidate solution of the following form:

$$f(X,P,K) = G(X,P,K) + \frac{1}{\overline{r}}. \tag{30}$$

Then, $G(X,P,K)$ satisfies the following homogeneous PDE:

$$\overline{r}G = \bar{\mu}_K K G_K + \bar{\mu}_X X G_X + \bar{\mu}_P P G_P + \frac{1}{2}\sigma_K^2 K^2 G_{KK} + \frac{1}{2}\sigma_P^2 P^2 G_{PP} + \tag{31}$$

$$\frac{1}{2}\sigma_X^2 X^2 G_{XX} + \rho_{KP}\sigma_K\sigma_P KPG_{KP} + \rho_{KX}\sigma_K\sigma_X KXG_{KX} + \rho_{XP}\sigma_X\sigma_P PXG_{XP}.$$

Next, we implement the change of variables

$$p = \log P,\ \ k = \log K,\ \ x = \log X, \tag{32}$$

thus leading to

$$\bar{r}G = (\bar{\mu}_K - \frac{1}{2}\sigma_K^2)G_k + (\bar{\mu}_X - \frac{1}{2}\sigma_X^2)G_x + (\bar{\mu}_P - \frac{1}{2}\sigma_P^2)G_p + \frac{1}{2}\sigma_K^2 G_{kk} + \frac{1}{2}\sigma_P^2 G_{pp} + \tag{33}$$

$$\frac{1}{2}\sigma_X^2 G_{xx} + \rho_{KP}\sigma_K\sigma_P G_{kp} + \rho_{KX}\sigma_K\sigma_X G_{kx} + \rho_{XP}\sigma_X\sigma_P G_{xp}.$$

Let $w \equiv k(\alpha - 1) + x - p$, and note that $w = \ln \omega$. Then, we have:

$$\begin{aligned}
G_p &= -G_w, \\
G_k &= (\alpha - 1)\, G_w, \\
G_x &= G_w, \\
G_{pp} &= G_{xx} = G_{ww}, \\
G_{kk} &= (\alpha - 1)^2\, G_{ww}, \\
G_{px} &= -G_{ww}, \\
G_{kx} &= (\alpha - 1)\, G_{ww}, \\
G_{kp} &= -(\alpha - 1)\, G_{ww}.
\end{aligned}$$

Now implementing the change of variables $w = k(\alpha - 1) + x - p$, we obtain a second-order ordinary differential equation for $G(w)$:

$$\frac{1}{2}\bar{\sigma}^2 G'' + \bar{\mu}G' - \bar{r}G = 0, \tag{34}$$

where $\bar{\mu}$ and $\bar{\sigma}^2$ are defined as

$$\begin{aligned}
\bar{\mu} &= \bar{\mu}_X - \bar{\mu}_P + (\alpha - 1)\,\bar{\mu}_K + \tfrac{1}{2}\left(\sigma_P^2 - \sigma_X^2 + (1 - \alpha)\,\sigma_K^2\right), \\
\bar{\sigma}^2 &= \sigma_X^2 + \sigma_P^2 + (\alpha - 1)^2\,\sigma_K^2 + 2\,(\alpha - 1)\,\rho_{KX}\sigma_K\sigma_X - 2\rho_{PX}\sigma_P\sigma_X - 2\,(\alpha - 1)\,\rho_{PK}\sigma_P\sigma_K.
\end{aligned} \tag{35}$$

By substituting (28) into (35), it is straightforward to verify that the definition of $\bar{\mu}$ here is consistent with the one in equation (11).

Our conjectured optimal investment policy will be characterized by a threshold value w^*, such that the firm does not invest if $w < w^*$. Therefore the firm's value must be always finite for $w < w^*$, and, in particular, it must approach zero as $w \to -\infty$ (the firm's value must approach zero as X_t goes to zero, since the current and expected future cash flows are proportional to X_t). Hence, we choose the following solution of the ODE (34)

$$G(w) = \frac{A}{\bar{r}} \exp\left(\lambda \left(w - w^*\right)\right), \tag{36}$$

where

$$\lambda = -\frac{\bar{\mu}}{\bar{\sigma}^2} + \sqrt{\left(\frac{\bar{\mu}}{\bar{\sigma}^2}\right)^2 + \frac{2\bar{r}}{\bar{\sigma}^2}}, \tag{37}$$

is the positive root of the quadratic equation

$$\lambda^2 + \frac{2\bar{\mu}}{\bar{\sigma}^2}\lambda - \frac{2\bar{r}}{\bar{\sigma}^2} = 0.$$

The positive root is chosen since $\lim_{w \to -\infty} G(w) = 0$. The constant of integration A and the no-investment boundary w^* can be found from the boundary conditions.

The boundary conditions at $w = w^*$ are the standard value-matching and smooth-pasting conditions that come from the second part of the variational inequality (27) (see, e.g., Dixit and Pindyck 1994, p.364). Implementing the same candidate solution (30) and the same changes of variables as in the no-investment region, the second part of the variational inequality becomes:

$$\left(\alpha - 1\right) G' + \alpha G = e^{-w} - \frac{\alpha}{\bar{r}}. \tag{38}$$

The first boundary condition (value-matching) states that the equality above has to be satisfied at w^*. The second (smooth-pasting) boundary condition can be obtained by differentiating (38) with respect to w:

$$e^{-w} = \left(\alpha - 1\right) G'' + \alpha G'.$$

The equality above has to be satisfied at w^*.

To summarize, we have:

$$\left(\alpha - 1\right) G'\left(w^*\right) + \alpha G\left(w^*\right) = e^{-w^*} - \frac{\alpha}{\bar{r}}, \tag{39}$$

$$e^{-w^*} = (\alpha - 1) G''(w^*) + \alpha G'(w^*). \tag{40}$$

Expressing e^{-w^*} from the first condition and substituting it into the second, we obtain:

$$(\alpha - 1) G''(w^*) + (2\alpha - 1) G'(w^*) - \alpha G(w^*) + \frac{\alpha}{\bar{r}} = 0. \tag{41}$$

Now we substitute the expression for $G(\cdot)$ from (36) to find A:

$$A = \frac{\alpha}{(\lambda + 1)(\lambda - \alpha\lambda - \alpha)}.$$

Finally, substituting (36) and the expression for A above into (39), we obtain:

$$\begin{aligned} e^{-w^*} &= \frac{\alpha}{\bar{r}} - (1 - \alpha) \frac{A}{\bar{r}} \lambda + \alpha \frac{A}{\bar{r}} \\ &= \frac{\alpha\lambda}{\bar{r}(1 + \lambda)}. \end{aligned}$$

It then follows that

$$\omega^* = e^{w^*} = \frac{\bar{r}(1 + \lambda)}{\alpha\lambda}.$$

We have now found both constants that were still to be identified in equation (36), A and w^*. The optimal investment process I_t^* is such that w^* serves as a reflecting barrier for w_t. The candidate solution is C^1 everywhere. Note further that the process w_t will never cross the threshold w^*, and our solution is C^2 for $w_t < w^*$. Since our solution is C^2 in the interior of its domain, it is indeed the value function for problem (25), see, e.g., Strulovici and Szydlowski (2015). Note further that since, by construction, $dI_t^* \geq 0$, the candidate control we have identified is monotonic and therefore has bounded variation. It is therefore indeed the optimal control (see Strulovici and Szydlowski 2015).

□

Proof of Proposition 2

We need to solve the following Bellman equation:

$$V\left(\hat{K}_t, X_t, P_t, g_{K,t}\right) = X_t g_{K,t}^{\alpha} \hat{K}_t^{\alpha} + P_t (1 - \delta) \hat{K}_t g_{K,t} + \tag{42}$$

$$+ \max_{\hat{K}_{t+1}} \left\{ \beta \cdot E\left[V\left(\hat{K}_{t+1}, X_{t+1}, P_{t+1}, g_{K,t+1}\right)\right] - P_t \cdot \hat{K}_{t+1} \right\}.$$

The first-order condition for $\hat{K}_{t+1}$ is:

$$\beta \cdot E\left[V_{\hat{K}_{t+1}}\left(\hat{K}_{t+1}, X_{t+1}, P_{t+1}, g_{K,t+1}\right)\right] = P_t.$$

Calculating $V_{\hat{K}}$ from (42) and substituting into the equation above, we obtain:

$$\beta \cdot E\left[\alpha X_{t+1} g_{K,t+1}^{\alpha} \hat{K}_{t+1}^{\alpha-1} + (1-\delta) P_{t+1} g_{K,t+1}\right] = P_t.$$

Therefore,

$$\alpha\beta E\left[g_X g_K^{\alpha}\right] X_t \hat{K}_{t+1}^{\alpha-1} + (1-\delta)\beta E\left[g_P g_K\right] P_t = P_t. \tag{43}$$

Let $a \equiv E\left[g_P g_K\right]$ and $b \equiv E\left[g_X g_K^{\alpha}\right]$. Then, from (43) we can find the optimal value of $\hat{K}_{t+1}$:

$$\hat{K}_{t+1} = \left(\frac{\alpha\beta b}{1-(1-\delta)\beta a}\right)^{\frac{1}{1-\alpha}} \left(\frac{X_t}{P_t}\right)^{\frac{1}{1-\alpha}}.$$

Therefore,

$$C_2 = \left(\frac{\alpha\beta b}{1-(1-\delta)\beta a}\right)^{\frac{1}{1-\alpha}}.$$

We will now look for a solution for the firm's equity value of the following form:

$$V\left(\hat{K}_t, X_t, P_t, g_{K,t}\right) = X_t g_{K,t}^{\alpha} \hat{K}_t^{\alpha} + P_t (1-\delta) \hat{K}_t g_{K,t} + C_3 X_t^{\frac{1}{1-\alpha}} P_t^{-\frac{\alpha}{1-\alpha}}, \tag{44}$$

where the constant C_3 is to be determined. Given the candidate solution in (44), we can calculate $E_t\left[V\left(\hat{K}_{t+1}, X_{t+1}, P_{t+1}, g_{K,t+1}\right)\right]$ as:

$$\begin{aligned} E_t\left[V\left(\hat{K}_{t+1}, X_{t+1}, P_{t+1}, g_{K,t+1}\right)\right] = {} & C_2^{\alpha}\left(\frac{X_t}{P_t}\right)^{\frac{\alpha}{1-\alpha}} X_t b + C_2 P_t \left(\frac{X_t}{P_t}\right)^{\frac{1}{1-\alpha}} P_t (1-\delta) a + \\ & + C_3 X_t^{\frac{1}{1-\alpha}} P_t^{-\frac{\alpha}{1-\alpha}} E\left[g_X^{\frac{1}{1-\alpha}} g_P^{-\frac{\alpha}{1-\alpha}}\right]. \end{aligned}$$

Let $c \equiv E\left[g_X^{\frac{1}{1-\alpha}} g_P^{-\frac{\alpha}{1-\alpha}}\right]$. Then, the expression above can be simplified to:

$$E_t\left[V\left(\hat{K}_{t+1}, X_{t+1}, P_{t+1}, g_{K,t+1}\right)\right] = X_t^{\frac{1}{1-\alpha}} P_t^{-\frac{\alpha}{1-\alpha}} \left\{bC_2^{\alpha} + C_2(1-\delta)a + C_3 c\right\}.$$

Now substituting the candidate solution (44) and the expression for $E_t\left[V\left(\hat{K}_{t+1}, X_{t+1}, P_{t+1}, g_{K,t+1}\right)\right]$

above into the Bellman equation (42) and simplifying, we get:

$$C_3 = \beta \left\{ b C_2^{\alpha} + C_2 \left(1-\delta\right) a + C_3 c \right\} - C_2.$$

From this we can find C_3:

$$C_3 = \frac{C_2}{1-\beta c} \left(\beta b C_2^{\alpha-1} + \left(1-\delta\right) \beta a - 1 \right).$$

Now note that the candidate solution in (44) can be written as:

$$\begin{aligned}
V\left(\hat{K}_t, X_t, P_t, g_{K,t}\right) &= X_t g_{K,t}^{\alpha} \hat{K}_t^{\alpha} + P_t \left(1-\delta\right) \hat{K}_t g_{K,t} + C_3 X_t^{\frac{1}{1-\alpha}} P_t^{-\frac{\alpha}{1-\alpha}} \\
&= CF_t + P_t \left(1-\delta\right) K_t + P_t I_t - P_t I_t + C_3 X_t^{\frac{1}{1-\alpha}} P_t^{-\frac{\alpha}{1-\alpha}} \\
&= CF_t - P_t I_t + P_t \hat{K}_{t+1} + \frac{C_3}{C_2} C_2 \left(\frac{X_t}{P_t}\right)^{\frac{1}{1-\alpha}} P_t \\
&= CF_t - P_t I_t + P_t \hat{K}_{t+1} \left(1 + C_1\right),
\end{aligned}$$

where $C_1 \equiv \frac{C_3}{C_2}$. It remains to simplify C_1 as follows:

$$\begin{aligned}
C_1 &= \frac{1}{1-\beta c} \left(\beta b C_2^{\alpha-1} + \left(1-\delta\right) \beta a - 1 \right) \\
&= \frac{1}{1-\beta c} \left(\frac{1 - \left(1-\delta\right) \beta a}{\alpha} + \left(1-\delta\right) \beta a - 1 \right) \\
&= \frac{\left(1-\alpha\right)\left(1 - \left(1-\delta\right) \beta a\right)}{\alpha \left(1-\beta c\right)}.
\end{aligned}$$

$\square$

Appendix B

In this section, we consider an application of our model to valuation of a firm that i) uses the historical cost model of IAS 16 to calculate the book value of its property, plant and equipment, and ii) recognizes write-downs as prescribed by IAS 36. Under IAS 36, an impairment loss has to be recognized if the carrying amount of an assets is greater than the maximum of its fair value less costs of disposal and value in use. The definition of value in use in IAS 36 is consistent with that used in our paper: it is equal to the present value of future cash flows expected to be derived from an asset. Different approaches to fair value measurement are defined in IFRS 13. According to this standard, the fair value of an asset should be measured as the price of an identical asset in an orderly transaction (if such price is observable) or using an appropriate valuation technique such as the the cost approach or the income approach.[37] Since in the setting with irreversible investments there is no secondary market for used capital goods, we will assume that the firm uses the cost approach to fair value measurement.[38]

Under IAS 36, when an impairment is recognized, the firm has to disclose whether the new book value of the asset reflects its fair value or value in use. In addition, past impairments are reversed if the recoverable amount increases, but such reversals cannot lead to a carrying amount greater than what it would have been had not impairment loss been recognized before. In this setting, we obtain the following bounds on the firm's equity value.

Corollary 3. *Assume the firm uses the cost approch to fair value measurement and recognizes impairments according to IAS 36. Then, the tightest bounds on the firm's equity value that hold almost surely conditional on $\mathcal{I}_t$ are:*
If $P_t dI_t > 0$,

$$V_t = \frac{CF_t}{\bar{r}} (1 + A)\,; \tag{45}$$

if a revaluation to replacement cost is recognized ($B_t = B_t^{rc}$),

$$V_t = \frac{CF_t}{\bar{r}} \left(1 + A \cdot \left[\frac{CF_t}{B_t \cdot \omega^*} \right]^{\lambda} \right); \tag{46}$$

[37] IFRS 13 explicitly states that value in use is not fair value (see paragraph 6).
[38] We also assume that the cost of disposal is immaterial.

if a revaluation to value in use is recognized,

$$\frac{CF_t}{\bar{r}}\left(1+A\cdot\left[\frac{\bar{r}}{\omega^*}\right]^{\lambda}\right)\leq V_t\leq\frac{CF_t}{\bar{r}}\left(1+A\right); \tag{47}$$

if $B_t > \frac{CF_t}{\bar{r}}$,

$$\frac{CF_t}{\bar{r}}\leq V_t\leq\frac{CF_t}{\bar{r}}\left(1+A\cdot\left[\frac{CF_t}{B_t\cdot\omega^*}\right]^{\lambda}\right); \tag{48}$$

otherwise

$$\frac{CF_t}{\bar{r}}\leq V_t\leq\frac{CF_t}{\bar{r}}\left(1+A\right). \tag{49}$$

We now briefly discuss the five cases described in the corollary above. First, if new investment is observed, $P_t dI_t > 0$, then the firm is at the investment threshold and its equity can be valued accordingly. If the firm recognizes a write-down (or reverses a past write-down) to fair value, then investors know that the current book value of assets reflects their replacement cost, and the firm's equity can be valued according to equation (46). If a write-down to value in use is recognized, then the new book value of assets is equal to

$$B_t = B_t^{viu} = \frac{CF_t}{\bar{r}}$$

and it has to be that $B_t \geq B_t^{rc}$ (otherwise a write-down to fair value would have been recognized). It follows that

$$V_t = \frac{CF_t}{\bar{r}}\left(1+A\cdot\left[\frac{CF_t}{B_t^{rc}\cdot\omega^*}\right]^{\lambda}\right)\geq\frac{CF_t}{\bar{r}}\left(1+A\cdot\left[\frac{CF_t}{B_t^{viu}\cdot\omega^*}\right]^{\lambda}\right)=\frac{CF_t}{\bar{r}}\left(1+A\cdot\left[\frac{\bar{r}}{\omega^*}\right]^{\lambda}\right),$$

and inequality (47) obtains.

Now assume that investors observe a book value of assets that is greater than the value in use:

$$B_t > \frac{CF_t}{\bar{r}}.$$

Then, it has to be that

$$B_t \leq B_t^{rc}$$

since otherwise a write-down would have been recognized. Therefore, in this case, investors know that

$$\frac{CF_t}{\bar{r}} \leq V_t \leq \frac{CF_t}{\bar{r}} \left(1 + A \cdot \left[\frac{CF_t}{B_t \cdot \omega^*}\right]^{\lambda}\right).$$

Lastly, when $B_t < B_t^{viu}$, the firm's replacement cost of assets can be arbitrarily close to zero or $\frac{CF_t}{\omega^*}$, and the tightest inequality that holds almost surely is given by (49).

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