Agency, Firm Growth, and Managerial Turnover

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Abstract

We study managerial incentive provision under moral hazard in an environment where growth opportunities arrive stochastically over time and taking them requires a change of management. The firm faces a tradeoff between the benefit of always having a manager able to seize new opportunities and the cost of incentive provision. The optimal dynamic contract may grant partial job protection whereby the firm insulates its managers from the risk of growth-induced dismissal and foregoes attractive opportunities when they come after periods of good performance. Moreover, the prospect of growth-induced turnover limits the firm's ability to rely on deferred pay—resulting in more front-loaded compensation. The empirical evidence for the U.S. is broadly supportive of the model's predictions. Industries with better growth prospects experience higher CEO turnover and use more front-loaded compensation.

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Introduction

When ownership and control are separated, firm performance depends crucially on having the right managers at the helm and incentivising them properly. Over time, changes in business conditions may call for a change of top management to seize new opportunities or overcome challenges faced by the firm. This, however, may complicate the task of incentivising incumbent managers. For instance, if managers anticipate that their tenure at the firm will be short, they will be reluctant to accept any form of deferred compensation, a standard feature of incentive contracts. Thus the firm may face a dilemma: by changing management to adapt to evolving business conditions, it may increase the costs of incentive provision.

To analyze this tension, this paper introduces the idea of *growth-induced* turnover into a dynamic moral hazard framework. Growth-induced turnover refers to the replacement of top management that is motivated by the need to have managers who possess the appropriate skill set and experience to lead the firm in its *current* circumstances. This may involve for instance adopting new production techniques, making acquisitions, or expanding into new markets. If the incumbent lacks the vision or skills necessary to implement such transformations, the appointment of new management is the only way for the firm to successfully pursue its course.[1] At the same time, proper dynamic incentive provision requires a combination of deferred compensation and a threat of dismissal following poor performance, both of which constitute agency costs. By introducing the possibility of managerial turnover for the sake of growth as well as for discipline, we show how these costs are affected. The main insight of the paper is that the prospect of growth-induced dismissal effectively increases managers' impatience, thus increasing agency costs and creating a general tendency to front-load compensation. In fact, the firm may actually be better off *ex ante* by committing to pass up otherwise attractive growth opportunities in some circumstances. More generally, our analysis delivers empirical predictions on the effects of a firm's growth prospects on managerial turnover and compensation which we show are broadly supported in the data.

In our model, a long-lived firm is run by a sequence of risk-neutral managers protected by limited liability. A moral hazard problem arises because while they are in charge, managers can divert cashflows for their own private benefit. The firm can fire the incumbent manager at any time and replace him at a cost. Fleeting growth opportunities arrive stochastically over time, and a change of management is needed to seize them. If the firm decides to take up an opportunity, it pays the costs associated with replacing the manager, and its size (or profitability) increases. A long-term incentive contract is signed between the firm and its successive managers at the time they are hired.

As in previous dynamic contracting studies, we show that optimal compensation and

[1] In some circumstances, a change of management may be required to avoid decay, rather than to actually grow—e.g., when by sticking with the *status quo*, the firm would fail to face up to a disruptive competitive threat. For instance, in his narrative of the battle waged in Canada around 1820 between the long-established Hudson Bay Company (HB) and its upstart rival North West Company (NW), Roberts (2004) recounts: "HB did respond to the threat, essentially by copying NW's new approach. It did so, however, *only after the leaders of the firm had been replaced by new ones who understood the nature of the threat and were not tied to the old ways that had worked so well for so long.*" (our emphasis)

turnover policies in this environment can be described in terms of a state variable that coincides with the agent's expected discounted compensation, referred to as his contractual 'promise'. The manager receives cash compensation only when his promise rises to reach an endogenous 'bonus threshold'. When the manager's promise lies below this threshold, cash compensation is deferred, and the promise is increased at a contractually specified rate plus a positive or negative adjustment based on the firm's current performance. If the firm suffers a sustained period of poor performance, the manager's promise can be lowered sufficiently to reach zero, the 'firing threshold', at which point the incumbent is replaced by a new manager who enters under a new contract with an exogenously given initial promise.

In contrast with other studies, the manager's contract in our framework is also contingent on the presence or not of a growth opportunity. If no growth opportunity becomes available, the manager continues his tenure so long as his promise stays above the firing threshold, and he is compensated with bonuses and performance-related changes in his promise as just described. If a growth opportunity arises and the firm takes it, the manager is replaced. However, not all growth opportunities are seized by all firms—even though they would be under first best. Specifically, we show that, depending on the characteristics of the firm and its environment, the optimal growth policy can be one of two types. For some firms, it is optimal to take all growth opportunities as they come. For other firms, it is optimal to forego opportunities that arise after periods of good performance, i.e., when the incumbent manager's promise is above a certain 'growth threshold'. We refer to these two different types of firms as *high-growth* and *low-growth* firms, respectively. In effect, optimal incentive provision in low-growth firms calls for some degree of job protection against the risk of growth-induced termination. Intuitively, the reason why job protection is granted after a spell of good cashflows is that losses due to agency problems are diminished after good performance, thus increasing the value of continuing with the incumbent manager net of the foregone benefit of growth. In high-growth firms, the benefit of growth always dominates.

Under the optimal contract, managerial compensation is affected by the possibility of growth-induced turnover through the drift of the manager's promise during his tenure. In the absence of growth opportunities, this drift would simply be equal to the manager's discount rate. The key novelty in our setup is that, whenever the firm stands ready to take an opportunity that might become available, the drift rate needs to be augmented to compensate the manager for the risk of growth-induced termination, with the drift modification depending on the arrival intensity of growth opportunities. This upwards adjustment of the drift when the firm stands ready to take a growth opportunity explains why firms with better growth prospects tend to have more front-loaded compensation. It also sheds light on why low-growth firms grant job protection when past performance has been good but not if it has been bad. A higher drift is indeed less costly to the firm after poor performance, i.e., when the manager's promise is close to the firing threshold, as it reduces the likelihood of a subsequent inefficient, disciplinary turnover.

Our analysis explicitly allows for the possibility of lump-sum payments, and we show that severance pay is suboptimal in our setting even in the case of growth-induced turnover. Indeed, it is always better for the firm to increase the incumbent's future promise conditional on him being retained, thereby making inefficient termination less likely in the future, than to give cash to a departing manager. However, we establish that an incoming manager may be given a 'signing bonus' if the required initial promise is high enough.

To derive these results on the second-best incentive contract, our approach roughly follows the same logic as in previous continuous-time analyses of dynamic moral hazard. First, we establish a state-space representation of long-term incentive contracts, where the state process coincides with the manager's promise as described above. Similar to other studies, no stealing is incentive compatible under a dynamic contract if the sensitivity of the manager's promise to reported cashflows is large enough. We then formulate the firm's contracting problem recursively in order to characterize the optimal incentive-compatible dynamic contract in the presence of stochastic growth opportunities. We show that the firm's size-adjusted value function can be characterized as the solution to a Hamilton-Jacobi-Bellman (HJB) equation that incorporates the possibility of growth-induced turnover in an intuitive way. This crucial step in the analysis is established through a verification theorem from which follow the main properties of optimal compensation and turnover policies. Based on the HJB, we also provide a characterization of the determinants of a firm's growth type. In particular, we show that low-growth firms tend to be those plagued with more severe agency problems. This finding suggests that better governance can work as an effective tool to promote economic growth.

Having characterized the optimal contract, we take full advantage of the dynamic nature of our model and elaborate further on its implications for the distribution of tenure length and for the timing of managerial compensation over tenure. These are complicated objects, as they are partly determined by the firm's type and the compensation and growth thresholds, all of which are endogenous. Optimal thresholds are given by the free-boundary points of rather involved free-boundary problems, and are shown to be implicitly defined as solutions to highly non-linear equations. We thus employ numerical simulations to discuss the impact of a firm's growth prospects on turnover and compensation under the optimal contract. The results illustrate the fact that firms with better growth prospects, in particular those with more attractive opportunities (i.e., holding their arrival intensity fixed), tend to have shorter tenure length and more front-loaded compensation. Holding everything else constant, average levels of managerial pay per period tend to be lower in firms with more modest growth prospects, but they also tend to grow more markedly over tenure, which contributes to a typically longer compensation duration for these firms.

Finally, we examine the data in light of the model. Merging data from CRSP, Compustat and ExecuComp for U.S. public companies over the period 1992-2013, we investigate empirically the links between firms' growth prospects, CEO turnover, and CEO compensation. Following an extensive literature in empirical corporate finance, we use average Q to capture the quality of firms' growth opportunities. Namely, we proxy the *ex ante* growth prospects of a firm at the time a new CEO is appointed by the lagged mean value of Q measured at the industry level. We first sort CEO episodes along this proxy and compare the distributions of tenure length and compensation duration across the highest and lowest quantiles of growth prospects. In line with the model predictions, we find that the CEOs of firms with better prospects tend to have shorter tenure and more front-loaded compensation. We confirm these findings by regression analysis. In a probit model, our proxy for firms' growth prospects is positively related to the likelihood of turnover, controlling for past performance. An increase in initial industry average Q by one standard deviation leads to an increase in the probability of turnover by 82 basis points. Since the unconditional frequency of CEO turnover in our sample is 8.4%, the effect is economically significant. We also find that the arrival of an opportunity, proxied by an increase in industry average Q since the beginning of a CEO's tenure, increases the probability of a turnover event, consistent with the notion of growth-induced turnover. Furthermore, the likelihood of turnover is less sensitive to the arrival of an opportunity when *ex ante* growth prospects were poor, in line with the prediction that firms with more modest growth prospects are more likely to insulate their managers from the risk of growth-induced turnover. Finally, we find that managerial pay tends to be lower in firms with worse growth prospects, and that the slope of the compensation profile over tenure years is more pronounced in such firms, which can be viewed as a manifestation of their greater reliance on compensation back-loading.

The idea that the pursuit of valuable growth opportunities by a firm may rely on a change of management is found in early contributions to the management literature, going back to Penrose (1959). More recently, Roberts (2004) studies a number of business cases where managerial limitations to firm growth play a prominent role and where a change of management is instrumental in unlocking the growth potential of a firm.[2] Bertrand and Schoar (2003) provide compelling evidence that managers indeed matter for firm performance and that they differ in their management styles. Bennedsen et al. (2012) further report that CEO effects are particularly important in rapidly growing environments. Building on the idea that firm productivity is determined by the quality of the match between the skill set of the manager and the *current* circumstances of the firm, Eisfeldt and Kuhnen (2013) analyze a competitive assignment model of CEO turnover where the skills demanded by the firm are subject to random shocks. In a similar vein, Jenter and Lewellen (2014) extend the standard Bayesian learning model of CEO turnover (e.g., Harris and Holmström (1982)) by allowing the quality of the firm-CEO match to vary over time. In contrast with our work, these papers abstract from agency issues and incentive considerations which occupy centre stage in our analysis.

Our paper relates to a large body of work that applies the tools of dynamic contracting to the study of the firm in the presence of agency conflicts.[3] In particular, Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007a), He (2008), Biais et al. (2010, 2013), Philippon and Sannikov (2011), and DeMarzo et al. (2012) investigate the link between moral hazard and firm growth when the firm can grow with the incumbent. Our main theoretical contribution is to focus instead on growth-induced turnover and its interactions with incentive provision. To the extent that the optimal contract in our setting is contingent on the realization of observable shocks, our work also bears some similarity with Piskorski and Tchistyi (2010) and Li (2015). More specifically, our framework builds on the continuous-time cash diversion model of DeMarzo and Sannikov (2006),[4] which we

[2]Also in the management literature, Chen and Hambrick (2012) document that, in turnaround situations, troubled companies substantially improve performance when they replace incumbent CEOs who are poorly suited to the conditions at hand with new ones who are well matched to those conditions.

[3]For seminal contributions to the literature on dynamic moral hazard, see Rogerson (1985) and Spear and Srivastava (1987) in discrete time, as well as Holmström and Milgrom (1987) and Sannikov (2008) in continuous time. Recent applications to the study of CEO turnover and compensation include among others, Spear and Wang (2005), Hoffman and Pfeil (2010), He (2012), Edmans et al. (2012), and Garret and Pavan (2012, 2015).

[4]See DeMarzo and Fishman (2007b) for a discrete-time version, and Biais et al. (2007) for an analysis of convergence from discrete to continuous time.

extend to incorporate the stochastic arrival of growth opportunities. From a technical point of view, our contributions are as follows. First, we introduce an additional source of uncertainty beyond the Brownian cashflow shocks, which renders the derivation of the state-space representation of the contract and the proof of the verification theorem substantially more challenging. Second, in contrast with most of the literature, we consider a stationary environment where the firm's continuation value at the time of firing a manager is endogenous. Third, we explicitly allow for jumps in the cumulative compensation process, which enables us to assess the optimality of severance pay. Finally, our extensive analysis of the HJB equation and associated free-boundary problems allows us to derive explicit existence and uniqueness results, as well as comparative statics that are new to our setting.

The implications of our model and the evidence we provide are connected to a vast empirical literature on the determinants of turnover and compensation for top management.[5] The literature on CEO turnover has mostly focused on the link between turnover and performance, as recently exemplified by Jenter and Lewellen (2014) and Jenter and Kanaan (2015).[6] We find that, controlling for performance, firms' growth prospects also contribute to explain the likelihood of CEO turnover. In terms of managerial compensation, the model predictions are in line with Murphy (1999) who points out that pay packages often include a bonus system based on the firm's reported earnings in excess of a performance target. They also echoe Kaplan and Minton (2012) who discuss the coincidence of shorter CEO tenures and higher CEO pay in the time series. We add to existing empirical studies on CEO compensation by investigating how the profile of CEO pay over tenure relates to firms' growth prospects. The degree of reliance on deferred compensation has received relatively little attention in the literature so far. An exception is the analysis by Clementi and Cooley (2010) who exploit information on CEOs' holdings of stocks and stock options to construct a measure of deferred compensation. Gopalan et al. (2014) focus on the duration of a CEO's total compensation award in a given year based on information about the vesting periods of separate components in the package. Instead, we measure the duration of compensation received over the entire tenure of a CEO and we document that this measure varies negatively with the firm's growth prospects at the time the CEO is hired.

The rest of the paper proceeds as follows. Section 1 describes our modelling setup and derives the state-space representation of long-term incentive contracts. Section 2 characterizes the optimal dynamic contract for high-growth and low-growth firms, as well as the determinants of firm type. Section 3 employs simulations to discuss the implications of our model for turnover and the timing of compensation. Section 4 presents the empirical evidence. Section 5 concludes.

1 The Model

We consider a firm run by a sequence of managers protected by limited liability. The firm and its managers are risk-neutral, with discount rates r and ϱ, respectively. The firm's

[5]For surveys of the literature on CEO compensation and on managerial incentive packages more generally, see for instance Murphy (1999, 2013).

[6]Early studies include, among others, Coughlan and Schmidt (1985), Warner et al. (1988), Weisbach (1988), Kim (1996), and Denis et al. (1997).

operations generate a stream of instantaneous cashflows $\Phi_t \, dY_t$, where Φ_t denotes the size of the firm at time t, and the cumulative size-adjusted cashflow process $Y = \{Y_t\}$ follows

$$dY_t = \mu \, dt + \sigma \, dZ_t, \qquad \mu, \sigma > 0,$$

where $Z = \{Z_t\}$ denotes a standard one-dimensional Brownian motion. The firm starts with unit size ($\Phi_0 = 1$) and can later expand. At any point in time, two conditions must be met for the firm to expand: (i) it must have a growth opportunity, and (ii) it must hire a new manager to take up the opportunity. Growth opportunities arrive sequentially, independently of cashflow shocks, and the waiting time for the arrival of the next opportunity is exponentially distributed with parameter q. If not taken immediately, an opportunity is lost and no further growth is possible until a new one arrives.

The assumption that the firm cannot grow without a change of management is central to our analysis. This assumption captures circumstances where value creation requires specific managerial skills to carry out radical transformations of the firm, and the incumbent does not have the ability to realize the firm's growth potential. For convenience, we model value enhancement as a discrete change in firm size that scales up the distribution of cashflows. Namely, we assume that when it expands, the size of the firm increases by a factor $1 + \gamma > 1$. Firm growth, when it occurs, is the result of bringing in a new manager able to take advantage of newly available opportunities—thereby achieving a permanent increase in expected cashflows.[7]

The second main feature of the model is a standard agency problem arising from the fact that, while running the firm's operations, managers can divert cashflows. The residual cashflow received by the firm is $\Phi_t \, (dY_t - dA_t)$, where $A = \{A_t\}$ denotes the cumulative size-adjusted amount of 'stealing'.[8] Managers enjoy a private benefit $\lambda \in (0, 1]$ for each unit of diverted cashflow, so that λ measures the severity of moral hazard.

The firm has deep pockets and can cover negative cashflows, as well as the costs associated with managerial compensation and turnover. Thus the firm's decisions are not driven by financing constraints. A manager hired to run the firm at size Φ_t receives expected discounted compensation $\bar{w}\Phi_t$, and the cost of replacing him is $\kappa\Phi_t$, where $\bar{w}, \kappa > 0$ are given constants. The assumption that compensation is increasing in firm size is motivated by the idea that larger operations require a different and less wide-spread skill set.[9] Likewise, we assume that turnover costs, which include indirect costs such as disruption of on-going business, are increasing in firm size.[10] The stronger assumption that compensation and turnover costs are both proportional to size is made to preserve tractability.[11] The continuation value of a departing manager is normalized to zero.

[7]This may or may not involve an increase in the fixed assets of the firm. If it does, future scaled cashflows should be thought of as net of the financing cost of capital investments.

[8]The stealing strategy A chosen by the manager is adapted to the Brownian filtration and has continuous sample paths. Given that Y is continuous, any jump in A would be immediately detected by the firm.

[9]Empirically, executive pay is indeed positively correlated with firm size, both over time and across firms, as documented by Kostiuk (1990), Murphy (1999), and Gabaix and Landier (2008).

[10]Estimates of the various costs associated with CEO transitions (including disruption costs) for mid-cap companies are roughly twice as large as those borne by small-cap companies, and less than half the costs borne by large-cap companies (Chief Executive Magazine, Nov/Dec 2008).

[11]Similar proportionality assumptions ensuring size homogeneity can be found in, e.g., Biais et al. (2010), and DeMarzo et al. (2012).

We further assume throughout the paper that

$$\varrho > r, \tag{1}$$
$$r > q\gamma, \tag{2}$$
$$\frac{\gamma\mu}{r} > \kappa + (1+\gamma)\bar{w}, \tag{3}$$

and we refer to parameter values that satisfy these conditions, along with the ones previously imposed in this section, as *permissible*. Condition (1) requires that managers are more impatient than the firm.[12] Condition (2) imposes that the average growth rate when the firm takes all growth opportunities is smaller than the firm's discount rate, which ensures finite valuation. Finally, together with (2), condition (3) implies that in the absence of moral hazard, it would be optimal for the firm to take all growth opportunities—as we next establish.

1.1 First-Best Policy

The first-best policy can be characterized as follows. First, the optimal compensation policy involves giving to a manager a size-adjusted transfer $\bar{w}$ *at the outset* of his tenure. Indeed, since managers are more impatient than the firm, deferring compensation would affect firm value negatively. Second, in order to save on replacement and hiring costs, managerial turnover never occurs if not for the sake of taking a growth opportunity. Third, the optimal growth policy involves either taking all growth opportunities or never taking any. If the firm takes all opportunities, its expected discounted profit V^* satisfies

$$V^* = -\bar{w} + \mathbb{E}\left[\int_0^\tau e^{-rt}dY_t + e^{-r\tau}\left[(1+\gamma)V^* - \kappa\right]\right],$$

where τ is the random arrival time of the first growth opportunity. Solving for V^* under the assumption that τ is exponentially distributed with parameter q yields

$$V^* = \frac{\mu - q\kappa}{r - q\gamma} - \frac{r+q}{r-q\gamma}\bar{w}.$$

If instead the firm foregoes all opportunities, its expected discounted profit is given by

$$-\bar{w} + \mathbb{E}\left[\int_0^\infty e^{-rt}dY_t\right] = -\bar{w} + \frac{\mu}{r}.$$

It is straightforward to see that conditions (2) and (3) are sufficient for the inequality

$$V^* > \max\left\{-\bar{w} + \frac{\mu}{r},\ 0\right\}$$

to hold true, which guarantees that it would be optimal for the firm to take all growth opportunities under first best.

[12] This assumption is standard in the dynamic contracting literature (e.g., DeMarzo and Sannikov (2006), Biais et al. (2007, 2010), and DeMarzo et al. (2012)). The wedge in discount rates rules out indefinitely postponing payments to managers.

1.2 Long-Term Incentive Contract

We now turn to the case where managers can divert cashflows and stealing is not observable by the firm. The firm enters into a long-term contract with each manager at the time of his hiring, and both parties fully commit to the terms of the contract. A contract specifies circumstances upon which the manager will be dismissed, including those when the firm will take a growth opportunity, as well as the manager's pay over the course of his tenure based on the information that will become available to the firm over time. In particular, the arrival of a growth opportunity is assumed to be perfectly observable and contractible. To fix ideas and simplify the exposition, we initially restrict our attention to the contract with the first manager. Readers interested in the technical aspects of the sequential contracting environment are referred to Appendix A.

First, we discuss how dismissal and compensation are determined for a given stealing strategy A chosen by the manager. The information accruing to the firm over time comes from observing the cumulative reported cashflows $\hat{Y} = Y - A$, as well as the arrival of growth opportunities. We denote by $\mathcal{F}_t$ the information gathered by the firm up to time t, which includes information about the occurrence of growth opportunities. We denote by $\hat{\mathcal{F}}_t \subseteq \mathcal{F}_t$ the information coming only from the history of reported cashflows up to time t.

Dismissal of the manager can occur for two distinct reasons in our setting. First, the manager can be sacked after a history of poor reported cashflows. Indeed, committing *ex ante* to fire the incumbent after poor reported performance can be used by the firm as a device to incentivize him not to steal. Second, the manager can be replaced in order to take a growth opportunity that becomes available. Hence turnover is partly governed by the firm's *growth policy*, which determines the firm's response to the potential arrival of a growth opportunity. This policy is modeled by an $(\hat{\mathcal{F}}_t)$-progressively measurable process $G = \{G_t\}$ taking values in $\{0, 1\}$, with $G_t = 1$ indicating that the firm stands ready to take a growth opportunity at time t, and $G_t = 0$ indicating that it does not.[13] Importantly, by controlling G, the firm effectively determines the instantaneous intensity of growth-induced dismissal, which is equal to qG_t at time t. In view of these observations, the random time τ at which the manager is fired can thus be represented as[14]

$$\tau = \tau_{\mathrm{d}} \wedge \tau_{\mathrm{g}},$$

where τ_{d} denotes an $(\hat{\mathcal{F}}_t)$-stopping time and the random time τ_{g} satisfies[15]

$$\mathbb{P}\Big(\tau_{\mathrm{g}} > t \,\Big|\, \hat{\mathcal{F}}_t\Big) = \exp\left(-\int_0^t qG_s\, ds\right). \tag{4}$$

[13]Note that G_t is set by the firm without knowledge of whether an opportunity arises or not at time t. In Appendix C, we show that randomization of the growth decision, i.e., $G_t \in (0, 1)$, is suboptimal.

[14]We use the notation $x \wedge y$ to denote the minimum of x and y, i.e., $x \wedge y = \min\{x, y\}$.

[15]The left-hand side of (4) denotes the probability that the manager has not been dismissed for the sake of growth by time t, conditional on the history of reported cashflows up to time t. The right-hand side captures the fact that the instantaneous intensity of growth-induced dismissal at time $s \leq t$ is qG_s. When the firm stands ready to take all growth opportunities, setting $G \equiv 1$, the probability that the manager survives the threat of growth-induced termination up to time t is given by $\exp(-qt)$, reflecting the fact that the arrival of opportunities is exponentially distributed with parameter q.

In the event that $\tau = \tau_{\mathrm{d}}$, the manager is replaced for the sake of incentive provision, which we refer to as *disciplinary turnover*. When instead $\tau = \tau_{\mathrm{g}}$, the manager is dismissed for the sake of growth, which we refer to as *growth-induced turnover*.

Compensation to the manager over the course of his tenure is captured by an $(\hat{\mathcal{F}}_t)$-adapted cumulative compensation process $C = \{C_t\}$. Limited liability implies that C is increasing. A positive jump ΔC_t represents a lump-sum payment at time t.[16] In particular, ΔC_0 and $\Delta C_{\tau_{\mathrm{d}}}$ denote a signing bonus and severance pay upon disciplinary dismissal, respectively. To capture severance pay upon growth-induced turnover, we introduce a separate $(\hat{\mathcal{F}}_t)$-progressively measurable process $S = \{S_t\}$. The amount of severance received by a manager dismissed for the sake of growth is given by $S_{\tau_{\mathrm{g}}}$.

Now considering the set $\mathcal{A}$ of all possible stealing strategies, a contract can thus be viewed as a function mapping each stealing strategy $A \in \mathcal{A}$ to a collection

$$C = C(A), \quad S = S(A), \quad G = G(A), \quad \text{and} \quad \tau_{\mathrm{d}} = \tau_{\mathrm{d}}(A),$$

as just described. Such mapping should be consistent across stealing strategies in the sense that any given history of reported cashflows should result in the same compensation and termination outcomes, independently of the underlying combination of true cashflows and stealing that gave rise to that observed history. The contract space $\mathcal{G}$ identifies with the set of all functionals $\Gamma : A \mapsto \big(C(A), S(A), G(A), \tau_{\mathrm{d}}(A)\big)$ on $\mathcal{A}$ that satisfy this requirement.

1.3 The Firm's Problem

Given a contract Γ and a stealing strategy A, the manager's expected discounted payoff at the time of his hiring is given by

$$M(\Gamma, A) = \mathbb{E}\left[\int_{[0,\tau[} e^{-\varrho t}\left(dC_t + \lambda\, dA_t\right) + e^{-\varrho\tau}\Big(\Delta C_{\tau_{\mathrm{d}}}\mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + S_{\tau_{\mathrm{g}}}\mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big)\right].$$

For a given contract Γ, a stealing strategy A is said to be *incentive compatible* if it maximizes the manager's payoff. We refer to a contract as *admissible* if it is such that (i) no stealing is incentive compatible, and (ii) the manager's expected discounted payoff under no stealing is $\bar{w}$. Formally, the subset $\mathcal{G}_{\mathrm{a}}$ of admissible contracts includes all contracts $\Gamma \in \mathcal{G}$ such that

$$M(\Gamma, 0) = \sup_{A \in \mathcal{A}} M(\Gamma, A) \qquad \text{and} \qquad M(\Gamma, 0) = \bar{w}.$$

Given an admissible contract Γ, the firm's expected discounted profit at $t = 0$ is

$$\begin{aligned} F(\Gamma) = \mathbb{E}\bigg[& \int_{[0,\tau[} e^{-rt}\left(\mu\, dt - dC_t\right) \\ & + e^{-r\tau}\Big([V_{\mathrm{d}} - \Delta C_{\tau_{\mathrm{d}}} - \kappa]\mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + [V_{\mathrm{g}} - S_{\tau_{\mathrm{g}}} - \kappa]\mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big)\bigg], \end{aligned} \tag{5}$$

[16] We assume that C is right continuous with left limits and $C_{0-} = 0$, therefore $\Delta C_t = C_t - C_{t-}$ and $\Delta C_0 = C_0$. Further technical details on the modelling of long-term incentive contracts are given in Appendix A.1.

where V_{d} and V_{g} denote the firm's continuation values after dismissal of the first manager (for disciplinary reasons or upon growth, respectively), which we endogenize later in Section 2.[17] The firm's problem is to find an admissible contract that maximises its expected discounted profit. Formally, the firm's objective is to find Γ^* such that

$$F(\Gamma^*) = \sup_{\Gamma \in \mathcal{G}_{\mathrm{a}}} F(\Gamma).$$

1.4 Admissible Dynamic Contracts

As observed in previous work on dynamic moral hazard, the challenge of analyzing this type of environment comes from the complexity of the contract space and from the difficulty of evaluating the agents' incentives in a tractable way. In this section, we build on the approach of DeMarzo and Sannikov (2006), Sannikov (2008), and Biais et al. (2007, 2010), and consider a state-space representation of incentive contracts. Under no stealing, the state variable in this representation should coincide with the manager's expected payoff. As a preliminary step, we therefore characterize the process followed by

$$M_t = \mathbb{E}\left[\int_{]t,\tau[} e^{-\varrho(s-t)} dC_s + e^{-\varrho(\tau-t)}\left(\Delta C_{\tau_{\mathrm{d}}} \mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + S_{\tau_{\mathrm{g}}} \mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\right) \,\Big|\, \mathcal{F}_t\right], \quad t < \tau,$$

which corresponds to the manager's expected future payoff at time $t < \tau$ when he refrains from stealing.

Lemma 1. *For any given any contract $\Gamma \in \mathcal{G}$, there exists a process $\beta = \{\beta_t\}$ such that*

$$dM_t = \big[\varrho M_t + qG_t(M_t - S_t)\big]dt - dC_t + \sigma\beta_t \, dZ_t, \qquad \textit{for } t < \tau. \tag{6}$$

Proof. See Appendix B.1.[18]

The presence of the diffusion term in the dynamics of the agent's expected payoff is very natural. Since compensation and dismissal policies are contingent on the history of reported cashflows, the evolution of the manager's expected future payoff under a long-term incentive contract is sensitive to currently reported cashflows. The process β can precisely be interpreted as the sensitivity induced by a long-term contract. Since reported cashflows coincide with true cashflows when the manager refrains from stealing, the stochastic evolution of the manager's expected payoff under no stealing M_t is directly driven by the true cashflow shocks dZ_t.

[17] In Appendix A.2, we provide an expression for the firm's value at $t = 0$ for a given sequence of admissible contracts. In particular, when the same admissible contract Γ is offered to all managers, we show that the firm's size-adjusted expected discounted profit $F(\Gamma)$ satisfies (5) with $V_{\mathrm{d}} = F(\Gamma)$ and $V_{\mathrm{g}} = (1+\gamma)F(\Gamma)$. Note that we make the usual assumption that, if no stealing is incentive compatible, the manager does not steal. We restrict our attention to contracts that implement no stealing, which is standard in the literature when moral hazard is modelled as a cash diversion problem (e.g., see DeMarzo and Sannikov (2006)).

[18] In our model, uncertainty is not only driven by the Brownian cashflow shock but also by the stochastic arrival of growth opportunities. As a result, the derivation of (6) does not simply rely on the martingale representation theorem, as in the standard martingale approach developed by Sannikov (2008), but also on a "change of filtration" formula and other techniques borrowed from the credit risk literature.

In light of Lemma 1, we consider *dynamic contracts* whose implementation is driven by a state process $W = \{W_t\}$ that evolves as

$$dW_t = \big[\varrho W_t + qG_t(W_t - S_t)\big]\,dt - dC_t + \beta_t\,(d\hat{Y}_t - \mu\,dt). \tag{7}$$

Along with compensation and growth policies, a dynamic contract specifies the sensitivity β of the state variable W to the reported cashflows. Importantly, since the dynamics of the state variable are driven by processes that are either observed or controlled by the firm, its evolution over time can be tracked by the firm. While growth-induced turnover is jointly determined by the growth policy and the random arrival of opportunities, disciplinary dismissal occurs when the state process W hits zero, namely,

$$\tau_{\mathrm{d}} = \inf\{t \geq 0:\ W_t = 0\}. \tag{8}$$

Noting that $d\hat{Y}_t - \mu\,dt = -dA_t + \sigma\,dZ_t$, it is straightforward to see that, when the manager refrains from stealing, the dynamics of the state process become

$$dW_t = \big[\varrho W_t + qG_t(W_t - S_t)\big]\,dt - dC_t + \sigma\beta_t\,dZ_t \tag{9}$$

and therefore mirror (6). Indeed, when the manager refrains from stealing, the value taken by the state variable at any time during his tenure does coincide with his expected future compensation under the contract, as stated in the following lemma.

Lemma 2. *Consider a dynamic contract with termination occurring at time $\tau = \tau_{\mathrm{g}} \wedge \tau_{\mathrm{d}}$, where τ_{g} satisfies (4) and τ_{d} is defined by (8) where W follows (7) for some initial condition $W_{0-} = w_{\mathrm{init}} > 0$. Then the manager's expected future payoff at time $t < \tau$ if he refrains from stealing is equal to W_t, namely,*

$$W_t = \mathbb{E}\left[\int_{]t,\tau[} e^{-\varrho(s-t)}dC_s + e^{-\varrho(\tau-t)}\Big(\Delta C_{\tau_{\mathrm{d}}}\mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + S_{\tau_{\mathrm{g}}}\mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big)\,\Big|\,\mathcal{F}_t\right] \tag{10}$$

on the event $\{t < \tau\}$. Moreover, if $\beta \geq \lambda$, it is optimal for the manager not to steal.

Proof. See Appendix B.3.

Equation (10) confirms that the state process W under a dynamic contract can be interpreted as the manager's expected future payoff if he refrains from stealing, which we shall refer to as the manager's *promise*. Lemma 2 also establishes an incentive compatibility condition.[19] The condition is intuitive: since he enjoys a private benefit λ per unit of diverted cash, incentivizing the manager not to steal requires that his promise increases by at least λ for each extra unit of reported cashflow, namely, $\beta \geq \lambda$. A dynamic contract is admissible if it satisfies this condition along with the initial promise condition $W_{0-} = \bar{w}$. Since $\beta \geq \lambda > 0$ under an admissible dynamic contract, (7) and (8) imply that inefficient disciplinary turnover occurs as the result of *poor* reported cashlows.

Before proceeding further, it is important to observe that, relative to the environment considered in DeMarzo and Sannikov (2006), the introduction of growth-induced turnover

[19]This result extends the incentive compatibility condition derived in DeMarzo and Sannikov (2006) to our environment with growth-induced turnover.

affects the dynamics of the agent's promise in a substantial way. The key difference lies in the drift of the promise, which in our setup is equal to $\varrho W_t + qG_t(W_t - S_t)$ instead of simply ϱW_t. The reason for this difference is that, whenever the manager is put at risk of being fired for the sake of growth (i.e., whenever $G_t = 1$), he needs to be 'compensated' for the loss that he would incur in case a growth opportunity arises. The potential loss corresponds to the difference $(W_t - S_t)$ between the manager's current promise and the amount of severance pay that he would receive if replaced for the sake of growth,[20] while the chances of incurring such loss are determined by the instantaneous intensity of growth-induced dismissal qG_t. Compensation for the risk of growth-induced termination comes in the form of an augmented drift, which translates into a faster increase of the manager's promise in states of the world where no growth opportunity materializes. In other words, the law of motion for the agent's promise is modified in our setup in such a way that the *promise keeping* condition remains satisfied.

2 Optimal Dynamic Contract

Having characterized the set of admissible dynamic contracts, we reformulate the firm's optimization problem as a stochastic control problem. We denote by $V(\phi, w)$ the firm's value function, which gives the firm's expected discounted profit at a given current size ϕ and for a given current size-adjusted promise w to the incumbent manager. The firm's value function satisfies the recursive dynamic programming equation

$$\begin{aligned} V(\phi, w) = \sup_{C,S,G,\beta} \mathbb{E}\Big[\phi \int_{[0,\tau[} e^{-rt}(\mu dt - dC_t) - \phi e^{-r\tau}\Big(\Delta C_{\tau_{\mathrm{d}}} \mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + S_{\tau_{\mathrm{g}}} \mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big) \\ + e^{-r\tau}\Big(-\kappa\phi + V(\phi, \bar{w})\mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + V\big((1+\gamma)\phi, \bar{w}\big)\mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big)\Big], \end{aligned} \tag{11}$$

where $\tau = \tau_{\mathrm{d}} \wedge \tau_{\mathrm{g}}$, subject to the incentive compatibility constraint $\beta \geq \lambda$, and subject to (4), (8) and (9) with initial condition $W_{0-} = w$. In this formulation, C and S denote the manager's size-adjusted cumulative compensation and size-adjusted severance upon growth, respectively. Since cashflows, turnover costs and initial promises are all proportional to firm size, it follows that firm value itself is homogenous in size, namely,

$$V(\phi, w) = \phi V(1, w) =: \phi v(w). \tag{12}$$

In particular, stationarity and size homogeneity imply that the firm offers the same dynamic contract to all successive managers. Using (11) and (12), the size-adjusted value function $v(w)$ is determined along with the optimal contract by

$$\begin{aligned} v(w) = \sup_{C,S,G,\beta} \mathbb{E}\Big[\int_{[0,\tau[} e^{-rt}(\mu dt - dC_t) - e^{-r\tau}\Big(\Delta C_{\tau_{\mathrm{d}}} \mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + S_{\tau_{\mathrm{g}}} \mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big) \\ + e^{-r\tau}\Big(-\kappa + v(\bar{w})\mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + (1+\gamma)v(\bar{w})\mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big)\Big] \end{aligned} \tag{13}$$

[20] We later show that under the optimal contract, a manager receives no severance when dismissed for the sake of growth. The potential loss upon growth-induced termination is therefore equal to W_t under the optimal contract.

subject to the same constraints as above. The following proposition is central to our characterization of the optimal dynamic contract.

Proposition 1. *Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a concave C^2 function that satisfies the Hamilton-Jacobi-Bellman (HJB) equation*

$$\max\left\{ \frac{\sigma^2\lambda^2}{2} u''(w) + \varrho w u'(w) - r u(w) + \mu \right.$$
$$\left. + q\left[(1+\gamma)u(\bar{w}) - \kappa + w u'(w) - u(w)\right]^+ ,\ -u'(w) - 1 \right\} = 0, \tag{14}$$

with boundary condition

$$u(0) = u(\bar{w}) - \kappa. \tag{15}$$

Also, suppose that $\lim_{w\downarrow 0} |u'(w)| < \infty$ and $u'(w) = -1$ for some $w < \infty$. Then the function u identifies with the value function v defined by (13), namely, $v(w) = u(w)$ for all $w \geq 0$. Moreover, the optimal dynamic contract satisfies Properties 1-5 listed below.

Proof. See Appendix C.

We rely on Proposition 1 to construct the firm's value function and solve for the optimal dynamic contract. As observed in previous work on dynamic moral hazard, the concavity of the value function is related to the fact that a change in w affects firm value not only directly by increasing the amount of compensation owed to the manager, but also via its impact on the likelihood of disciplinary turnover. Indeed, by reducing the prospect of a costly disciplinary turnover, an increase in the agent's promise by one dollar effectively costs less than one dollar to the firm. Moreover, since the probability of disciplinary turnover is higher after poor performance, the reduction in agency costs induced by a marginal increase in the agent's promise is larger for low values of w. This is what gives rise to concavity.

2.1 Optimality Properties

We now turn to the properties satisfied by the optimal dynamic contract, as implied by Proposition 1. These impose restrictions on the cashflow sensitivity, on the compensation policy, and on the growth policy. The first two properties are standard. In particular, they also hold in the absence of growth opportunities, and are derived in that context by DeMarzo and Sannikov (2006), and Biais et al. (2007).[21]

Property 1. *The optimal contract has sensitivity to reported cashflows $\beta = \lambda$.*

The fact that the incentive compatibility constraint should hold as an equality ($\beta = \lambda$) is related to the concavity of the value function. Intuitively, reducing the volatility of the manager's promise as much as possible while satisfying incentive compatibility is optimal for the firm because it lowers the probability that the promise hits zero, which would result in *ex post* inefficient disciplinary turnover.

[21] From a broader perspective, the claims stated in Property 2 are inherited from standard features of the solution to singular stochastic control problems such as the one given by (13); see Beneš et al. (1980) and Karatzas (1983) for early references.

Property 2. *The optimal compensation policy is such that the manager receives transfers only if his current promise w is at least w_c. The compensation threshold w_c satisfies*

$$v'(w_c) = -1.$$

This property can be explained heuristically by observing that, at any instant, the firm has the option to make an immediate transfer to the manager and continue optimally. Hence, the inequality $v(w) \geq -\varepsilon + v(w - \varepsilon)$ holds for any transfer ε, which implies $v'(w) \geq -1$. When the manager's current promise w is such that $v'(w) > -1$, deferring compensation is optimal. By concavity of the value function, this happens when w is below the point w_c that satisfies $v'(w_c) = -1$. In this case, the manager receives no compensation until his promise reaches the compensation threshold. If $\bar{w} > w_c$, the manager receives a signing bonus $\Delta C_0 = \bar{w} - w_c$ when appointed, and his promise later remains in the interval $[0, w_c]$.[22]

Property 3. *The optimal compensation policy involves no severance payment, namely, $\Delta C_{\tau_d} = 0$ and $S = 0$.*

Severance pay is suboptimal in our setup, even in the case of growth-induced termination. The reason is that, rather than give cash to a departing manager, the firm is always better off increasing the promise of the incumbent conditional on him being retained, which has the benefit of reducing the likelihood of inefficient turnover later on.[23] It is worthwhile to note that the no-severance result upon growth-induced dismissal relies crucially on the assumption that the arrival of a growth opportunity is contractible.[24]

Property 4. *It is optimal for the firm to stand ready to take a growth opportunity if and only if the manager's current promise w is such that*

$$(1 + \gamma)v(\bar{w}) - \kappa + wv'(w) \geq v(w). \tag{16}$$

Condition (16), which we shall refer to as the *growth optimality* condition, determines the circumstances under which growth-induced turnover can occur. The inequality reveals that the optimal growth policy does not just rely on a comparison between the *status quo* continuation value $v(w)$ and the continuation value upon growth $(1 + \gamma)v(\bar{w}) - \kappa$. The extra term $wv'(w)$ accounts for the fact that putting the manager at risk of being fired if a growth opportunity arrives requires to compensate him in the form of an augmented drift,

[22]Technically, the agent's promise W is *reflected* at w_c by the cumulative compensation process. A rigorous construction of this process is provided in Appendix C (see Theorem C-2). One way in which the introduction of stochastic growth opportunities and growth-induced turnover modifies the firm's pay policy is by affecting the value of the optimal threshold w_c, as shown in Section 2.3 for the high-growth case.

[23]By the same logic, severance pay would be suboptimal in a simpler setting with exogenous random exit of the manager. We are grateful to an anonymous referee for making this observation.

[24]In an earlier version of this paper, we analyzed the case in which the availability of a growth opportunity is privately observed by the incumbent manager and showed that, in that case, severance upon growth-induced dismissal arises as part of the optimal contract (See Anderson et al. (2012), Section 6). This result is reminiscent of Eisfeldt and Rampini (2008) and Inderst and Mueller (2010), although in our model severance is used to incentivize the incubent to reveal *good* news. Malenko (2013) also considers an environment with privately observed investment opportunities. Severance pay upon *disciplinary* dismissal arises in the setup analyzed by He (2012) with risk-averse agent and private savings.

as discussed in Section 1.4. When the firm's value function is decreasing at the current value of the agent's promise (i.e., $v'(w) < 0$), this higher drift constitutes a cost. If this cost is high relative to the potential gains from growth, so that $(1+\gamma)v(\bar{w})-\kappa+wv'(w)-v(w) < 0$, it is optimal for the firm to insulate the incumbent manager from the risk of being replaced, and thus forego growth opportunities when they become available. We refer to this possibility as contractual *job protection*.

Property 5. *If partial job protection arises as part of the optimal contract, the firm foregoes growth opportunities if the manager's promise w is above $w_{\rm g}$, where the growth threshold $w_{\rm g}$ satisfies*

$$w_{\rm g} = \sup\big\{w \geq 0 : \ (1+\gamma)v(\bar{w}) - \kappa + wv'(w) - v(w) \geq 0\big\} < w_{\rm c}.$$

This property indicates that, if some degree of job protection arises as part of the optimal incentive contract, managers are shielded from the risk of growth-induced turnover after good performance. Intuitively, the benefit of retaining the incumbent, net of the foregone gains from growth, is increasing in w because losses due to moral hazard under the incumbent are dimished after good performance.[25]

2.2 Two Types of Firms

In light of our discussion of Properties 4 and 5, two configurations can arise. In the first one, the growth optimality condition (16) holds for all values of the manager's promise $w \in [0, w_{\rm c}]$. We refer to firms falling into this configuration as *high-growth firms*. In such firms, managers are fully exposed to the risk of being fired for the sake of growth, and the instantaneous rate of growth-induced turnover is always equal to q. Over the course of the manager's tenure, the firm keeps track of the evolution of

$$dW_t = (\varrho + q)W_t\,dt - dC_t + \lambda\,(d\hat{Y}_t - \mu\,dt), \quad W_{0-} = \bar{w},$$

where transfers dC reflect the manager's promise W at the endogenous compensation threshold $w_{\rm c}$. Transfers to the manager can be interpreted as bonuses indexed on reported performance.[26] The manager is dismissed when a growth opportunity arises or when W hits zero, whichever comes first.

By contrast, in the second possible configuration, the growth optimality condition does not hold everywhere on the interval $[0, w_{\rm c}]$, and some degree of job protection is part of the optimal contract. We refer to firms falling into the latter configuration as *low-growth firms*. The contract offered by a low-growth firm specifies, along with a compensation

[25]In other words, the net benefit of exposing a manager to the risk of growth-induced termination is decreasing in the manager's promise, which can be seen from the fact that $(1+\gamma)v(\bar{w}) - \kappa + wv'(w) - v(w)$ is decreasing in w, by concavity of v. Our finding that, in some firms, growth may only occur after poor performance is in contrast with the result obtained in setups where the firm can grow through investment with the incumbent (see, e.g., DeMarzo and Fishman (2007a)). In such settings, growth is positively related to past performance because the return on investment is higher after good cashflows, also due to a reduction in agency costs.

[26]In both types of firms, transfers to the manager are increasing in reported cashflows net of the expected level of cashflows. This feature of the contract is qualitatively in line with the use of bonus systems based on reported earnings in excess of a performance target, as documented in Murphy (1999).

threshold $w_{\rm c}$, a growth threshold $w_{\rm g} < w_{\rm c}$. Over the course of the manager's tenure, the firm keeps track of

$$dW_t = \big[\varrho + q\mathbf{1}_{[0,w_{\rm g}]}(W_t)\big]W_t\,dt - dC_t + \lambda\,(d\hat{Y}_t - \mu\,dt), \quad W_{0-} = \bar{w},$$

where transfers dC reflect W at $w_{\rm c}$. The manager is dismissed if a growth opportunity arises at a time when $W_t \leq w_{\rm g}$, or when W hits zero, whichever comes first. Consistent with our discussion of Property 5, the optimal contract in low-growth firms commands that, whenever the manager's promise is above the growth threshold $w_{\rm g}$, the firm foregoes any growth opportunity that becomes available.[27]

It might not seem obvious that the low-growth configuration could ever be optimal, given that condition (3) guarantees that foregoing growth opportunities is inefficient under first best. The intuition is that, in the presence of moral hazard, a firm faces this *ex ante* tradeoff: a policy of always standing ready to pursue growth by appointing a new, more suitable manager has the advantage of producing higher expected cashflows, but it entails increased early termination risk for incumbent managers and a higher cost of incentive provision during their tenure. Indeed, putting a manager at risk of being replaced for the sake of growth effectively makes him more impatient, as revealed by the fact that the agent's 'effective' discount rate is augmented from ϱ to $\varrho + q$. Therefore, insuring managers against early termination risk can make it cheaper to incentivize them. In low-growth firms, the resolution of the tradeoff between efficient turnover and the cost of incentive provision gives rise to an 'interior' solution, whereby the optimal contract allows for job protection after good performance.[28]

2.3 High-Growth Firms

In this section, we further characterize the optimal contract offered by a high-growth firm. To this end, we consider the free-boundary problem that consists in finding a free-boundary point $w_{\rm c}$ and a function u that satisfies the ODE

$$\frac{\sigma^2\lambda^2}{2}u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] = 0 \tag{17}$$

in the interval $(0, w_{\rm c})$, is given by

$$u(w) = u(w_{\rm c}) - (w - w_{\rm c}), \quad \text{if } w > w_{\rm c}, \tag{18}$$

and satisfies the boundary conditions

$$u(0) = u(\bar{w}) - \kappa, \quad u'(w_{\rm c}) = -1 \quad \text{and} \quad u''(w_{\rm c}) = 0. \tag{19}$$

[27]The manager being partially shielded from the risk of growth-induced turnover might be described as an endogenous form of 'entrenchment'. We do not use this word because it more commonly connotes actions taken by a manager to make his replacement costly. A number of recent papers explore frameworks very different from ours where they establish conditions under which managers are protected from termination. See, e.g., Atkeson and Cole (2008), Casamatta and Guembel (2010), and Garrett and Pavan (2012).

[28]A third possible configuration involves fully isolating the managers from the risk of growth-induced termination, which corresponds to (16) being violated for all values of $w \in [0, w_{\rm c}]$. However, we show in Appendix E.5 that this 'no-growth' policy can only be optimal if $v(\bar{w}) < 0$, so that the firm would rather not operate. We do not expand further on this case in the remainder of our analysis.

Proposition 2. *Given any permissible values of* $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ *in* $\mathbb{R}^9$*, there exists a unique solution* (u, w_c) *to the free-boundary problem defined by (17)–(19). The function* u *is* C^2 *and concave, and satisfies the HJB equation*

$$\max\left\{\frac{\sigma^2\lambda^2}{2}u''(w) + (\varrho + q)wu'(w) - (r + q)u(w) \right.$$
$$\left. +\mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big],\ -u'(w) - 1\right\} = 0.$$

Furthermore, the following statements hold true:

i. The set of permissible parameter values for which u *satisfies (16) for all* $w \in [0, w_\mathrm{c}]$*, and therefore the HJB equation (14), has non-empty interior in* $\mathbb{R}^9$*.*

ii. There exists a unique $\bar{w}_\dagger = \bar{w}_\dagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$ *such that*

$$\bar{w} \le \bar{w}_\dagger \ \Leftrightarrow w_\mathrm{c} > \bar{w} \qquad \textit{and} \qquad \bar{w} = \bar{w}_\dagger \ \Leftrightarrow \ w_\mathrm{c} = \bar{w}.$$

iii. If $\bar{w} = \bar{w}_\dagger$*,* u *satisfies (16) for all* $w \in [0, w_\mathrm{c}]$ *and the HJB equation (14) if and only if*

$$\gamma\mu \ge r\kappa + (r + \gamma\varrho)\bar{w}. \tag{20}$$

Proof. See Appendix E.2.

In view of Proposition 1 and the general properties of the solution to the free-boundary problem established in Proposition 2, statement (i) implies that, for a large set of permissible parameter values, the firm is of the high-growth type. For such parameter values, the firm's size-adjusted value function and the optimal compensation threshold are given by the solution to the free-boundary problem (17)–(19). Figure 1 illustrates the firm's value function and the optimal compensation threshold in the high-growth configuration for particular parameter values.

[FIGURE 1 HERE]

Statement (ii) sheds light on how the optimal compensation policy plays out at the start of a manager's tenure in a high-growth firm. Depending on the value of the starting promise $\bar{w}$, three scenarios can arise. When the initial promise is relatively low, in the sense that $\bar{w} < \bar{w}_\dagger$, the compensation threshold is optimally set above the initial promise ($w_\mathrm{c} > \bar{w}$). In this scenario, a newly hired manager does not receive any pay for some time until the effect of the positive drift $\varrho + q$, possibly combined with good cashflow realizations, finally takes his promise up to the compensation threshold w_c. In contrast, if the starting promise is high enough ($\bar{w} > \bar{w}_\dagger$), a manager receives a signing bonus $\Delta C_0 = \bar{w} - w_\mathrm{c} > 0$ when appointed. In the knife-edge case $\bar{w} = \bar{w}_\dagger$, the manager starts receiving compensation immediately after taking office.

Statement (iii) provides an explicit condition on exogenous parameter values for the firm to be a high-growth type. Condition (20) suggests that high-growth firms tend to be the ones that are more productive (high μ) or have better opportunities (high γ). Our next proposition gives further insight on the characteristics of high-growth firms.

Proposition 3. *Consider any permissible values of* $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ *in* $\mathbb{R}^9$ *such that* $\bar{w} = \bar{w}_\dagger$ *and condition (20) holds with equality. A marginal increase in* λ *or* κ*, or a marginal decrease in* q*, leads condition (20) to fail.*

Proof. See Appendix E.3.

In view of statement (*iii*), this result suggests that high-growth firms also tend to be characterized by not-too-severe moral hazard, low turnover costs, and more frequent growth opportunities. Although the statement of Proposition 3 is not generic, extensive numerical analysis confirms that these findings hold more generally.[29]

Finally, we characterize the determinants of the optimal compensation threshold in high-growth firms with the following proposition.

Proposition 4. *Consider* $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ *in the interior of the set of permissible parameter values for which the firm is a high-growth type.*

i. The optimal compensation threshold w_c *is increasing in* κ*, and independent of* μ *and* γ*.*

ii. If the parameter values are initially such that $\bar{w} = \bar{w}_\dagger$*, then a marginal increase in* λ *or* σ *leads to an increase in* w_c*, whereas a marginal increase in* q *leads to a reduction in* w_c*.*

Proof. See Appendix E.4.

As the severity of moral hazard, the volatility of cashflows, or the cost of managerial replacement increases, the compensation threshold is raised to reduce the likelihood of inefficient turnover. On the other hand, an increase in the arrival rate of growth opportunities, by increasing the manager's effective discount rate, results in a lower compensation threshold.[30] Holding the dynamics of the manager's promise constant, a lower (resp., higher) compensation threshold results in more front-loaded (resp., back-loaded) compensation. We further discuss the implications of our model for the timing of compensation in Section 3.2.

2.4 Low-Growth Firms

We now turn to the low-growth configuration. In light of Proposition 1 and Property 5, we consider the free-boundary problem that consists in finding two free-boundary points w_c and $w_g < w_c$ and a function u that satisfies the ODE

$$\frac{\sigma^2\lambda^2}{2}u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] = 0 \tag{21}$$

in the interval $(0, w_g)$, satisfies the ODE

$$\frac{\sigma^2\lambda^2}{2}u''(w) + \varrho wu'(w) - ru(w) + \mu = 0 \tag{22}$$

[29]The same comment applies to the results stated in Proposition 4.(ii).

[30]The result that the optimal compensation threshold w_c is unaffected by the mean size-adjusted cashflow μ differs from the one derived in DeMarzo and Sannikov (2006), where w_c is increasing in μ. The reason for this difference is that the firm's continuation value upon termination is exogenously given in their setup, whereas it is endogenously determined in ours.

in the interval $(w_{\mathrm{g}}, w_{\mathrm{c}})$, is given by

$$u(w) = u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}}), \quad \text{if } w > w_{\mathrm{c}}, \tag{23}$$

satisfies the boundary conditions given by (19), and satisfies the requirement that

$$u(w_{\mathrm{g}}) - w_{\mathrm{g}} u'(w_{\mathrm{g}}) = (1+\gamma)u(\bar{w}) - \kappa. \tag{24}$$

The analysis of this problem allows us to establish that, despite the assumption that foregoing growth opportunities is suboptimal under first best, the low-growth configuration can arise, as stated in the following proposition.

Proposition 5. *The set of permissible parameter values for which the solution u to the free-boundary problem defined by (19) and (21)–(24) is C^2 and concave, and satisfies the HJB equation (14), has non-empty interior in $\mathbb{R}^9$. For such parameter values, the firm is a low-growth type.*

Proof. See Appendix F.3.

In low-growth firms, the optimal contract is as described in Section 2.2, with growth and compensation thresholds given by the free-boundary points w_{g}, w_{c} of the free-boundary problem defined above.[31] Figure 2 depicts the firm value function along with the optimal thresholds in the low-growth configuration for particular parameter values. The figure also represents what the value of the firm would be if it were constrained to systematically take all growth opportunities as they come. The distance between the two curves on the figure illustrates the benefit that a low-growth firm derives from offering partial job protection to its managers, which can ultimately be traced back to a reduction in agency costs.

[FIGURE 2 HERE]

3 Empirical Implications

In this section, we further discuss the predictions of our model, focusing on the implications of growth-induced turnover. Namely, we investigate how a firm's growth prospects affect managerial turnover and pay. To do so, we rely on the analytical results of Section 2 and supplement those with numerical illustrations. Simulations of managerial spells under the optimal contract are used to characterize the distribution of the random dismissal time and aspects of the compensation process—as well as their dependence on model parameters. In Section 4, we draw on the insights gained from simulations to interpret some of our empirical findings through the lens of the model.

[31] In Appendix F.1, we derive three possible systems of highly non-linear equations that the points w_{g} and w_{c} should satisfy, depending on their location relative to $\bar{w}$; see in particular Problem F-0. Given the complexity of this problem, providing a complete characterization of its solution with a view to deriving a suitable solution to the HJB equation (14) with boundary condition (15) is beyond the scope of this paper.

3.1 Dismissal Probability and Tenure Length

In our setup, the probability of an incumbent manager being dismissed depends on the past performance of the firm under his tenure, but also on the availability of a growth opportunity and on the *ex ante* characteristics of the firm that affect the turnover policy. First, the likelihood of dismissal increases with poor performance—both because a string of bad cashflows can result in disciplinary turnover and because, in some firms, growth-induced turnover only occurs after poor performance. Second, holding performance and firm characteristics constant, the probability of dismissal also increases (at least weakly) upon arrival of a growth opportunity. Finally, the probability of turnover depends on firm characteristics, to the extent that these affect the contract specification and the degree of protection granted to the manager. In particular, firms with better growth prospects should show a higher turnover rate.

To see this last point, we consider two firms that are identical in every dimension except for the size (γ) of the growth opportunities they might receive. For the sake of illustration, we take as common parameter values across the two firms $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. In the firm with better growth prospects, we set $\gamma = 0.25$, while we set $\gamma = 0.10$ in the other firm.[32] The difference in the quality of growth opportunities faced by the two firms makes the former a high-growth type (i.e., a manager in this firm is never immune to the risk of growth-induced termination), and the latter a low-growth type (i.e., managers are protected from growth-induced turnover after good performance). The average annualized turnover rate in these two firms are 21.4% and 5.5%, respectively. Naturally, changes in growth prospects driven by the arrival rate (q) of growth opportunities have similar effects. To see this, we consider variations in q around the high-growth and low-growth baselines. For the high-growth firm, an increase in the frequency of growth opportunities from $q = 0.20$ to $q = 0.22$ causes the average turnover rate to rise to 23.2%, while setting $q = 0.18$ causes the turnover rate to drop to 19.4%. For the low-growth firm, the same variations in q cause the average turnover rate to rise to 5.7% or drop to 5.3%, respectively.

The model predictions on the likelihood of turnover translate into implications for the distribution of tenure length. Figure 3 depicts the cumulative distribution of tenure length in the two baseline examples. The probability distribution of tenure length for the low-growth firm first-order stochastically dominates the one for the high-growth firm, i.e., the probability of a manager reaching any given number of tenure years is higher in the low-growth firm. The median tenure length of a manager is 3.3 years in the firm with better growth prospects ($\gamma = 0.25$), whereas it is 12.6 years in the firm with poorer growth prospects ($\gamma = 0.10$). Changes in the quality of growth prospects driven by the arrival rate of growth opportunities q affect tenure length in a similar way: increasing the frequency of

[32] These parameter values are permissible (see conditions (1)–(3)). In particular, the firms' growth prospects are sufficiently attractive as to make taking all growth opportunities optimal in the absence of moral hazard. Discount rates r and ϱ, and the intensity rate q, are expressed on an annual basis. Given the normalization $\mu = 1$, parameters σ, κ, and $\bar{w}$ are effectively expressed in terms of annual mean cashflow. For given parameter values, we first determine the firm's type and the optimal contractual threshold(s) by solving numerically the free-boundary problems associated with the HJB equation (14)–(15), as described in Sections 2.3 and 2.4. The average turnover rate is then obtained from simulating the dynamics of the promise W under the optimal contract until dismissal for a very large number of managers.

growth opportunities from $q = 0.20$ to $q = 0.22$ causes the median time in office to drop to 3.0 years and 12.1 years, respectively; whereas switching to $q = 0.18$ causes that time to rise to 3.6 years and 13.1 years, respectively.

[FIGURE 3 HERE]

3.2 Compensation

Deferred compensation constitutes an essential feature of the optimal dynamic contract under moral hazard. It is well-understood that the degree of compensation back-loading should depend on the severity of moral hazard (λ), the cashflow volatility (σ), and the wedge between the manager's and the firm's discount rates ($\varrho - r$). In our setup, the extent of back-loading is also affected by the prospect of growth-induced turnover.[33]

We illustrate this aspect of our model by simulating managerial compensation under the optimal contract. First, we characterize the degree of back-loading using the notion of *compensation duration*. Namely, for a given sequence of bonuses received by a manager over his entire tenure, we compute the weighted average of the points in time when compensation is received—with weights equal to the fraction of the total discounted pay (using the agent's discount rate ϱ) received at each point in time.[34] Figure 4 depicts the cumulative distribution of realized compensation duration in the two baseline configurations introduced in the previous subsection. The relative position of the two distributions reflects the fact that, holding the level of expected discounted pay $\bar{w}$ constant across firms, compensation is more front-loaded in firms with better growth prospects. On average, compensation duration is 2.2 years in the high-growth firm, versus 4.8 years in the low-growth firm.[35]

[FIGURE 4 HERE]

To further characterize the impact of growth-induced turnover on the timing of compensation, we compute average compensation profiles over tenure in the high-growth and low-growth baselines. Namely, in any given firm, we compute the average pay received by managers as a function of time since appointment, conditional on still being in office. Figure 5 depicts these profiles in logs and illustrates key differences across firms, both in levels and in growth rates. In both firms, compensation profiles are increasing at the beginning of tenure and then reach a plateau. That the expected amount of compensation remains flat after a certain level of seniority reflects the fact that, in our model, the distribution of cashflows remains constant under a manager's tenure—since, by assumption, growth entails a change of management. That the expected amount of compensation increases at the beginning of tenure reflects the fact that the optimal contract in both firms specifies a

[33] Compensation under the optimal contract comes in the form of bonuses that the manager receives whenever his promise reaches the endogenous compensation threshold w_c (see Property 2 for a precise characterization). The analysis in Section 2 reveals that firms' growth prospects affect the timing of pay both through the drift of the manager's promise and the level of the compensation threshold.

[34] Compensation duration is formally defined as $\left(\int_0^\tau e^{-\varrho t}\, dC_t\right)^{-1} \int_0^\tau te^{-\varrho t}\, dC_t$, in line with the notion of bond duration used in interest rate risk management. In simulations, we use the discretized version of this expression.

[35] The high-growth and low-growth baselines only differ in terms of the size of growth opportunities (γ). An increase in the arrival rate of growth opporunities (q) also results in lower compensation duration.

compensation threshold w_c above the starting promise $\bar{w}$ so that, at the beginning of their tenure, only the most lucky managers receive a bonus, whereas later on, thanks to the upward drift of the promise W, managers can reach the compensation threshold even with an 'average' performance record. Comparing the profiles across firms, we make the following observations. First, at any level of seniority in the job, average compensation is higher in the firm with better growth prospects. This difference in compensation levels is required to guarantee that, despite different 'survival' probabilities, managers in both firms receive the same expected discounted payoff. Second, the firm with poorer growth prospects is characterized by more protracted and higher compensation growth over tenure—which is another manifestation of the greater extent of back-loading in such firms.

[FIGURE 5 HERE]

4 Empirical Evidence

In this section, we present empirical evidence that is consistent with the notion of growth-induced turnover and the model's implications illustrated in Section 3. We first investigate the empirical determinants of CEO turnover in the light of our model. We then explore the relation between the timing of CEO compensation and firms' growth prospects.

4.1 Data

Our empirical analysis relies on information on CEO tenure episodes in US public firms as reported in the Standard & Poor's ExecuComp database for the period 1992-2013.[36] Once we merge the ExecuComp sample with accounting information from Compustat and stock return data from CRSP, our sample comprises 4,386 CEO episodes. Out of these, 2,398 episodes cover the full tenure of the CEO from the year the CEO is appointed until the year of leaving post. The total number of CEO-year observations in our sample is 30,599.[37] The minimum number of firms covered in a given year is 925 in 1992, and the maximum is 1,647 in 2004.

Using information from Execucomp, we identify the beginning and end years of each completed CEO episode. The variable *TotTenure* is defined as the total number of years in which the CEO is running the firm. Within an episode, the variable *Turnover* is a dummy variable which equals 1 in the last year of the CEO's tenure and zero otherwise. We also use Execucomp to construct the variable *TotPay* defined as the total compensation awarded to a CEO in a given year.[38] Table 1 reports the summary statistics of our sample. In particular,

[36]The ExecuComp database covers all firms included in the S&P 500, MidCap, and SmallCap indexes. It would be interesting to extend the analysis to smaller public firms and private firms. Kaplan, Sensoy, and Stromberg (2008) document high turnover in the management teams of VC-backed private companies before going public.

[37]This includes observations for years prior to 1992 for episodes that start before 1992 and end after 1992 when we have information on the date at which the CEO was appointed.

[38]In common with many studies of CEO compensation (see, e.g., Murphy (1999, 2013)), we use total compensation in the year awarded. Further details on variable definitions and on the construction of our dataset are provided in Appendix G.

the average and median CEO tenure lengths are 6.4 years and 5 years, respectively, while the average annual turnover rate is 8.4%.

Our analysis centers on CEO turnover and the timing of CEO compensation in relation to the quality of a firm's growth prospects. Our empirical proxy for the growth prospects of a firm during a given CEO episode is based on the mean 'average Q' of all firms in the same industry. The use of average Q as a proxy for a firm's growth opportunities is standard in the empirical corporate finance literature.[39] We consider the arithmetic mean of average Q for all firms within the same 4-digit SIC industry group, denoted by *IndQ*. As a proxy for the quality of growth prospects during a given episode, we use the value of *IndQ* in the year before the CEO is appointed, which we denote by *IndQInit*. We interpret a higher value of *IndQInit* as capturing better *ex ante* growth prospects at the time a new CEO is hired.

Managerial turnover in our setup is also affected by the current availability of growth opportunities. Capturing the arrival of a growth opportunity in any given year during a CEO episode is challenging empirically. As a proxy, we construct the variable *RatioQ* defined as the ratio of *IndQ* in the firm's industry in any given year to *IndQInit*. A higher value of *RatioQ* will be observed if a larger proportion of firms within the industry have received a growth opportunity, implying that the firm itself is more likely to be facing such an opportunity.

To control for past performance within a CEO episode, we use the cumulative abnormal return of the firm over the past two years, denoted by *CAR*. We also consider return on assets in a given year (ROA) as an additional control for performance. Finally, we use the logarithm of the firm's total assets ($LnAssets$) to control for firm size.

[TABLE 1 HERE]

4.2 Determinants of CEO Turnover

We first examine the relation between turnover and the quality of growth prospects. As a first pass, Figure 6 depicts the cumulative distribution of CEO tenure length conditional on *ex ante* growth prospects proxied by *IndQInit*. The solid line plots the kernel estimate of the distribution for the upper twenty percent of CEO episodes ranked by *IndQInit*, whereas the dashed line corresponds to the bottom twenty percent. The cumulative distribution for the upper *IndQInit* sub-sample lies significantly above the one for the bottom *IndQInit* sub-sample.[40] That is, the likelihood that a CEO will not 'survive' beyond any number of years is higher for CEOs entering firms with good growth prospects than for CEOs entering firms with poor growth prospects, consistent with our model. Figure 6 constitutes the empirical counterpart to the simulation results depicted in Figure 3.

[FIGURE 6 HERE]

[39] The handbook by Eckbo et al (2001) surveys multiple studies in which average Q is used as a proxy for growth opportunities. This practice stems from Hayashi (1982), who derives sufficient conditions such that a firm's average Q coincides with its marginal product of capital. See Caballero (1997) and Bond and Van Reenen (2007) for surveys of the empirical literature assessing the links between average Q, marginal q, and investment.

[40] The two-sample Wilcoxon-Mann-Whitney test shows that the two distributions are significantly different from each other, with a p-value of zero. Similarly, the Kolmogorov-Smirnov test rejects the hypothesis that the two samples are drawn from the same distribution.

We further assess the implications of our model for managerial turnover by running a probit regression where the dependent variable is the *Turnover* indicator variable. The probit specification is as follows:

$$Prob(Turnover_{jt} = 1) = \Phi[\psi_0 + \psi_1 IndQInit_j + \psi_2 RatioQ_{jt} + \psi_3 CAR_{jt} + \boldsymbol{\alpha}' \boldsymbol{X}_{jt}],$$

where Φ is the standard normal cumulative distribution function, j denotes a CEO episode, t is calendar year, and $\mathbf{X}$ denotes a vector of control variables. In line with Jenter and Lewellen (2014b), we control for the return on assets in year t and the size of the firm in year t. Calendar year fixed effects are also included. Based on our analysis, we hypothesize that the coefficients on *IndQInit* and *RatioQ* should be positive, while the coefficient on CAR and ROA should be negative.

Table 2 summarizes the results of the probit regression. Column A reports the estimated coefficients of the probit model and their standard errors. All explanatory variables have the expected signs and are highly statistically significant. The coefficient on *IndQInit* is positive, in line with our model's prediction that turnover is more frequent in firms with better *ex ante* growth prospects. The coefficient on *RatioQ* is also positive, in accord with the idea that turnover is sometimes triggered by the arrival of growth opportunities. Finally, the coefficients on CAR and on ROA are negative, in line with the theoretical prediction that turnover is more likely after poor performance.[41]

Column B reports the implied marginal effects, which give the impact on the probability of turnover of a unit increase in an explanatory variable, when all variables are evaluated at the sample means. In Column C, the marginal effects are multiplied by the sample standard deviation of the corresponding explanatory variables. A one standard deviation increase in *IndQInit* is associated with an increase in the probability of turnover by 82 basis points. Similarly, a one standard deviation increase in *RatioQ* leads to a 67 basis point increase in the probability of turnover. Since the unconditional frequency of CEO turnover in our sample is 8.4%, these results support the view that the growth-related drivers of managerial turnover emphasized in this paper are economically significant. Of course, this is in addition to the disciplinary role of turnover, which we also find to be important. In our sample, a one standard error increase in past abnormal returns is associated with a drop in the probability of turnover by 2.2 percentage points.

[TABLE 2 HERE]

An additional implication of our model is a tendency for firms with relatively poor *ex ante* growth prospects to grant partial job protection to their CEOs. That is, our theory predicts that in such firms, a CEO is less likely to be dismissed for the sake of growth when an opportunity arises calling for his replacement. We explore the empirical validity of this prediction by evaluating the marginal effect of *RatioQ* on the probability of CEO turnover at different levels of *IndQInit*. According to the theory, the impact of *RatioQ* on turnover should be greater for firms with relatively better *ex ante* growth prospects, i.e., for higher values of *IndQInit*. Table 3 reports these differential marginal effects. The marginal effect of *RatioQ* is strictly positive at all levels of *IndQInit* and is indeed increasing in *IndQInit*.

[41] The results in Table 2 are unaffected when one-year or three-year cumulative abnormal returns are used as a measure of past performance, or when initial years of tenure are removed from the sample.

[TABLE 3 HERE]

4.3 Growth Prospects and CEO Compensation

In this subsection, we provide evidence on the empirical relation between CEO compensation and firms' growth prospects. A key insight from our model is that the managers of firms with better and more frequent growth opportunities should have more front-loaded compensation. We explore the data in two ways. First, we compute a measure of realized compensation duration for each CEO episode and investigate how it varies in relation to the growth prospects of the firm at the time the CEO was appointed. Second, we characterize how CEO compensation profiles over tenure years differ across firms with different growth prospects.

4.3.1 Compensation Duration

For a given CEO episode j lasting for N_j years, our measure of compensation duration, labelled *PayDuration*, is defined as:

$$PayDuration_j = \sum_{n=1}^{N_j} \frac{DiscPay_{j,n}}{\sum_{k=1}^{N_j} DiscPay_{j,k}} \times n, \tag{25}$$

where $DiscPay_{j,n} = TotPay_{j,n}/(1+\varrho)^n$ corresponds to the present value of the compensation received by the CEO in his n-th tenure year. Setting the discount rate ϱ at 10%, we find that, in the sub-sample of episodes over which this measure is computed, average CEO compensation duration is 3.4 years, while the median is 2.9 years.[42]

The empirical measure of pay duration defined by (25) is analogous to the one introduced in Section 3.2 to illustrate the implications of our model for managerial compensation. As an empirical counterpart to Figure 4, Figure 7 depicts kernel estimates of the cumulative distribution of *PayDuration* conditional on *ex ante* growth prospects proxied by *IndQInit*. The solid line pertains to the upper twenty percent of *IndQInit* in our sample, while the dashed line pertains to the bottom twenty percent. The cumulative distribution for the upper *IndQInit* sub-sample lies everywhere above the one for the bottom *IndQInit* sub-sample, which is consistent with our model's insight that firms with better growth prospects should have more front-loaded compensation.[43]

[FIGURE 7 HERE]

Table 4 reports the results from regressing *PayDuration* on *IndQInit* across CEO episodes, controlling for a combination of year and industry fixed effects.[44] The estimated coefficient on *IndQInit* is negative and highly significant across all specifications, confirming the negative relation between CEO compensation duration and firms' growth prospects.

[42]The measure of compensation duration is not very sensitive to the value of the discount rate. The conclusions from this subsection are robust to alternative values of ϱ.

[43]Again, the results from the two-sample Wilcoxon-Mann-Whitney test and from the Kolmogorov-Smirnov test both confirm that the difference between the two empirical distributions is statistically significant.

[44]In order to retain statistical power, we use one-digit SIC codes to control for industry fixed effects. Using a finer industry classification does not change the sign of the point estimate.

[TABLE 4 HERE]

4.3.2 Compensation Profile over Tenure

To further investigate how profiles of CEO pay over tenure vary across firms with different growth prospects, we estimate the regression equation

$$\begin{aligned} Ln(TotPay_{jt}) = \psi_0 + &\psi_1\, TenureYear_{jt} + \psi_2\, IndQInit_j + \\ &+ \psi_3\, IndQInit_j \times TenureYear_{jt} + \boldsymbol{\alpha}'\mathbf{X}_{jt} + \epsilon_{jt}, \end{aligned}$$

where j denotes a CEO episode, t is a calendar year, $TenureYear_{jt}$ (resp. $TotPay_{jt}$) denotes the number of years in tenure of CEO j in year t (resp. the total compensation received by the CEO in that year), and $\mathbf{X}$ is a vector of control variables. We control for past performance and firm size, as well as for calendar year fixed effects and industry or firm fixed effects. To estimate this regression, we use observations for all years in a CEO episode in which compensation data is available, resulting in many more observations than in the duration regressions reported in Table 4.

Our model suggests that firms with better *ex ante* growth prospects are characterized by a higher initial level of pay per period (i.e., ψ_2 positive), and slower growth in compensation over tenure years (i.e., ψ_3 negative). Table 5 summarizes our empirical findings for two alternative specifications, controlling for industry and firm fixed effects, respectively. The coefficients of interest are significant with the expected signs, and the results are very similar across both specifications. We also note that the coefficient on past abnormal returns is positive and significant, in line with the theoretical prediction that CEO pay is positively related to past performance.

[TABLE 5 HERE]

5 Conclusion

This paper introduces growth-induced turnover in a dynamic moral hazard framework and analyzes the interaction between this type of turnover and managerial incentive provision. In our model, growth opportunities arrive stochastically over time and the firm must appoint a new management to be able to seize them. Our analysis highlights the tradeoff that a firm faces between the benefit of always having at the helm a manager who is the right man for the job at hand and the cost of incentive provision. The key new insight is that exposing incumbent managers to the risk of growth-induced dismissal effectively increases their discount rate, thus increasing the cost of incentive provision. As a result, some firms find it optimal to provide some degree of job protection to their managers, at the cost of foregoing growth opportunities. Across firms or industries, a higher likelihood of growth-induced turnover translates into a greater tendency to front-load compensation. Our empirical findings are consistent with these predictions of the model.

An essential feature of our model is that non-disciplinary managerial turnover can be triggered by the firm contingent on the arrival of exogenous contractible shocks. In our setup, shocks correspond to the arrival of growth opportunities, and it is first-best efficient for the firm to replace the incumbent manager upon arrival of an opportunity. Our analysis could be applied to alternative forms of exogenous contractible shocks. First, transformative managerial change may also be important for firms in decline. For instance, a change of management may be required for a firm to respond to increased product market competition or to the threat of a disruptive new technology. Second, the firm may face opportunities to transform—through a change of management—that would bring gains that are too modest to outweigh the cost of implementing them, so that they would not be taken up under first best. Yet, in a second-best world, it may be optimal to take these inefficient opportunities when the agency costs associated with the current manager are high. We believe that a number of theoretical insights of the paper would carry through in these alternative settings, although the empirical implications would be quite different.

The existing empirical literature on managerial turnover and compensation has been mostly informed by two paradigms from the contracting literature—the moral hazard model in which pay and dismissal are used to incentivize the agent, and the learning model in which the principal learns over time about the unknown quality of the agent. In our view, transformative change can be another powerful driver of managerial turnover and compensation. We document the fact that industries with better growth prospects experience higher CEO turnover and rely on more front-loaded compensation schemes. These findings are consistent with the assumption of growth-induced turnover and the predictions of our model. Nonetheless, other theories may be consistent with these findings. Identifying the specific channel through which firms' growth prospects relate to CEO turnover and compensation deserves further empirical work.

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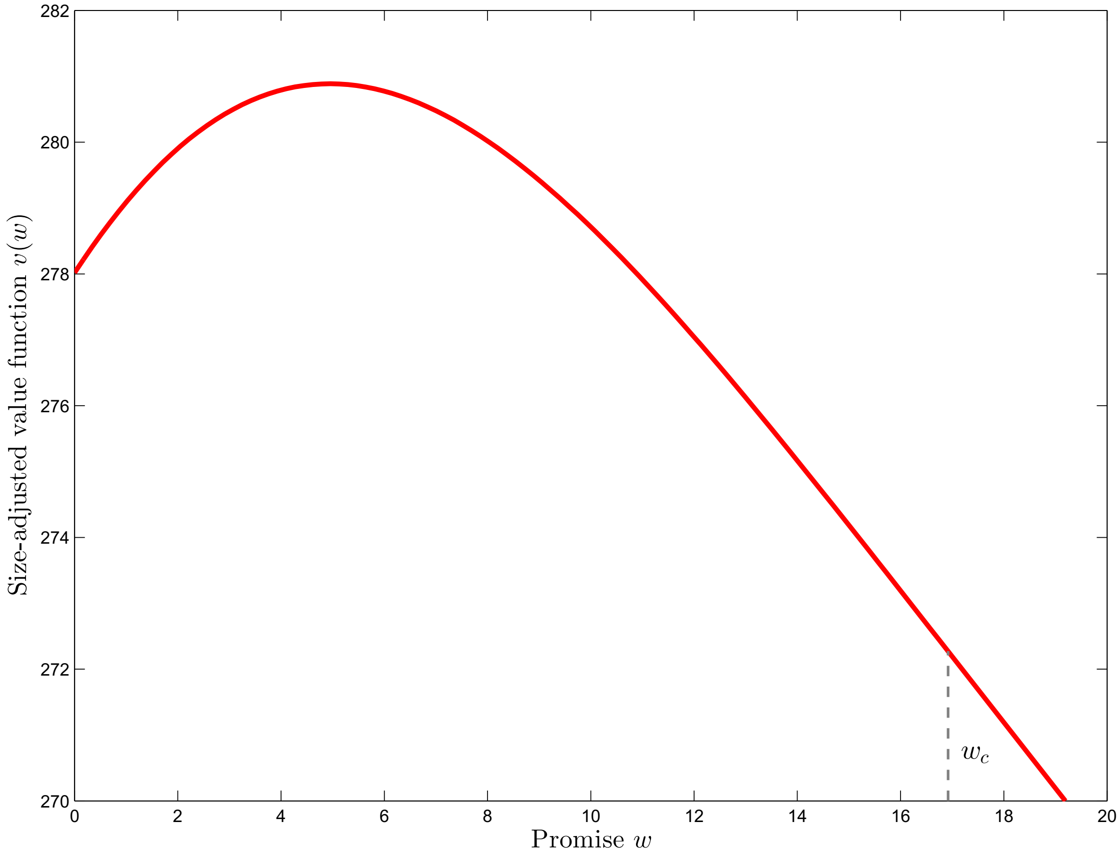

Figure 1: Value Function, High-Growth Firm

Notes: The figure depicts the firm's value function and the optimal compensation threshold w_c for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 10$, $\sigma = 12$, $q = 0.2$, $\gamma = 0.25$, $\lambda = 0.5$, $\kappa = 2$, $\bar{w} = 8$. The firm's value function and the compensation threshold are determined by solving the free-boundary problem defined in Section 2.3. The growth-optimality condition (16) holds for all values of the manager's promise.

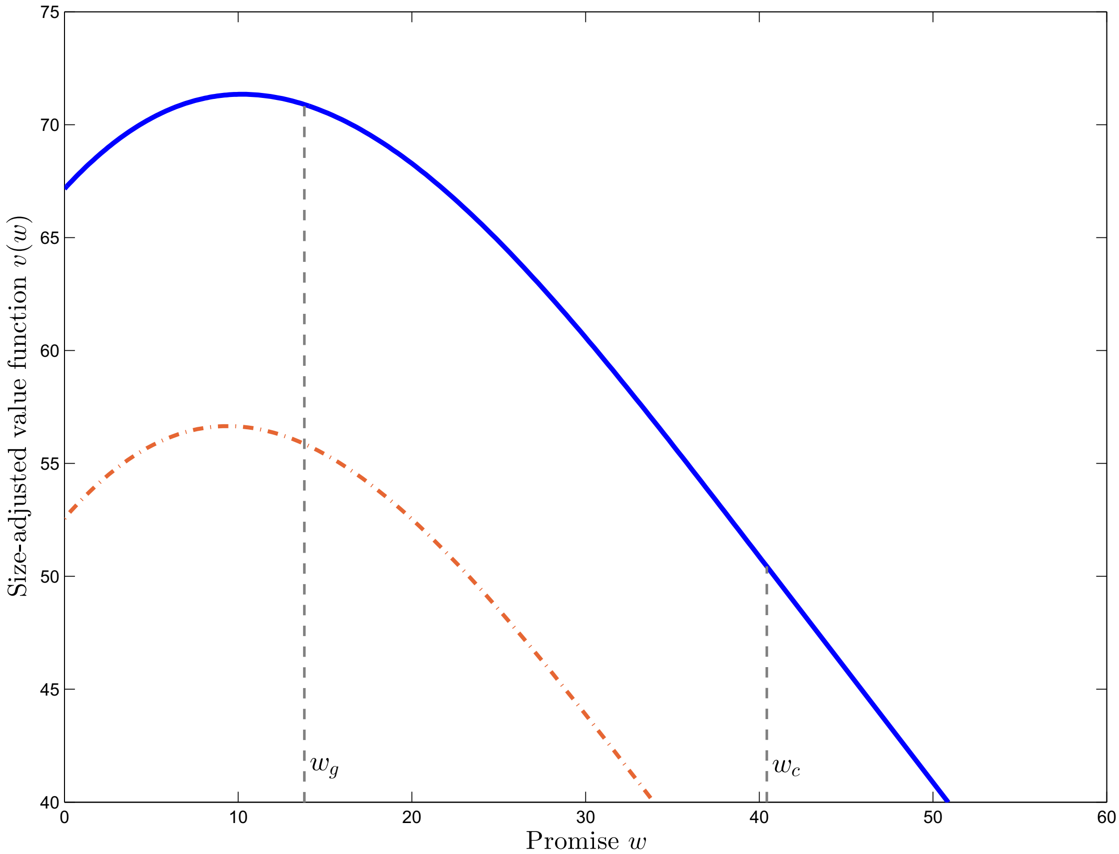

Figure 2: Value Function, Low-Growth Firm

Notes: The figure depicts the firm's value function (solid blue line), along with the optimal growth and compensation thresholds $(w_{\mathrm{g}}, w_{\mathrm{c}})$, for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 10$, $\sigma = 12$, $q = 0.1$, $\gamma = 0.1$, $\lambda = 1$, $\kappa = 4$, $\bar{w} = 8$. The firm's value function and the thresholds are determined by solving the free-boundary problem defined in Section 2.4. The growth-optimality condition (16) holds on $[0, w_g]$ but is violated for $w > w_{\mathrm{g}}$. The figure also represents (dashed orange line) what firm value would be if the firm were constrained to take all growth opportunities, with the compensation threshold optimally determined by the solution to the free-boundary problem defined in Section 2.3.

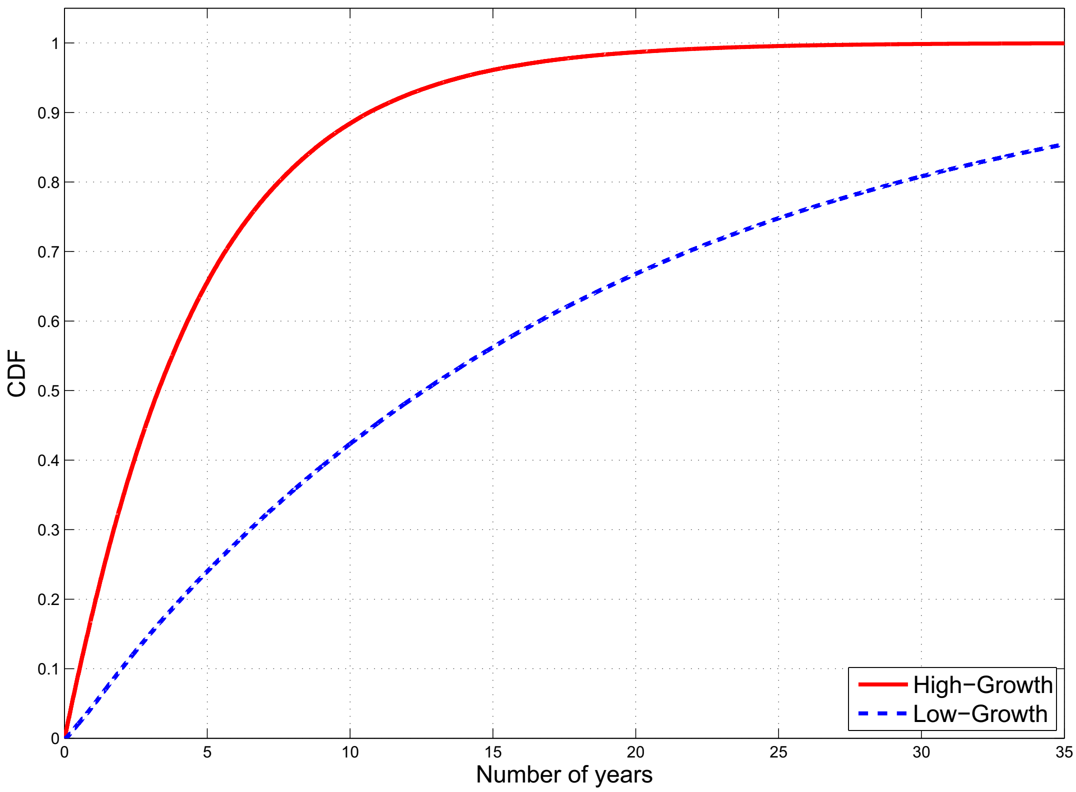

Figure 3: Distribution of Tenure Length: High-Growth vs. Low-Growth

Notes: The figure depicts model-implied cumulative distribution functions obtained from simulations for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\gamma = 0.25$ (high-growth) or $\gamma = 0.10$ (low-growth), $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. Optimal contractual thresholds are $w_{\rm c} = 1.24$ in the high-growth case, and $w_{\rm g} = 0.92$ and $w_{\rm c} = 1.36$ in the low-growth case.

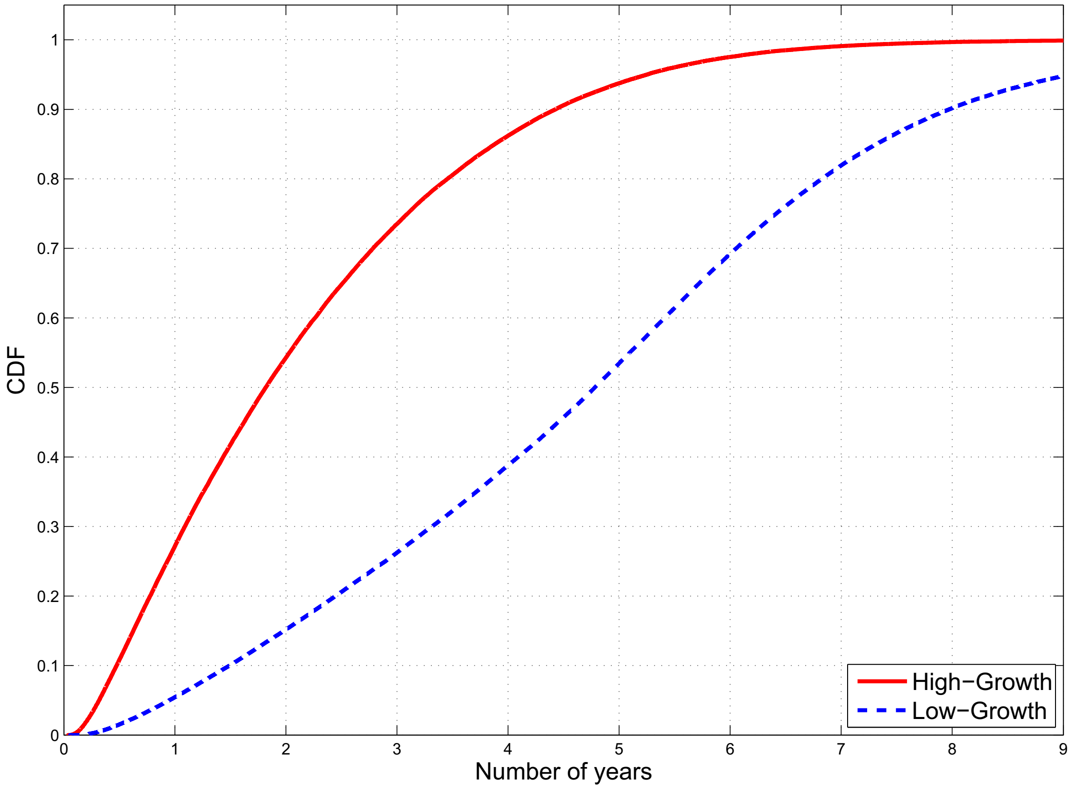

Figure 4: Distribution of Realized Compensation Duration: High-Growth vs. Low-Growth

Notes: The figure depicts model-implied cumulative distribution functions obtained from simulations for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\gamma = 0.25$ (high-growth) or $\gamma = 0.10$ (low-growth), $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. Optimal contractual thresholds are $w_c = 1.24$ in the high-growth case, and $w_g = 0.92$ and $w_c = 1.36$ in the low-growth case.

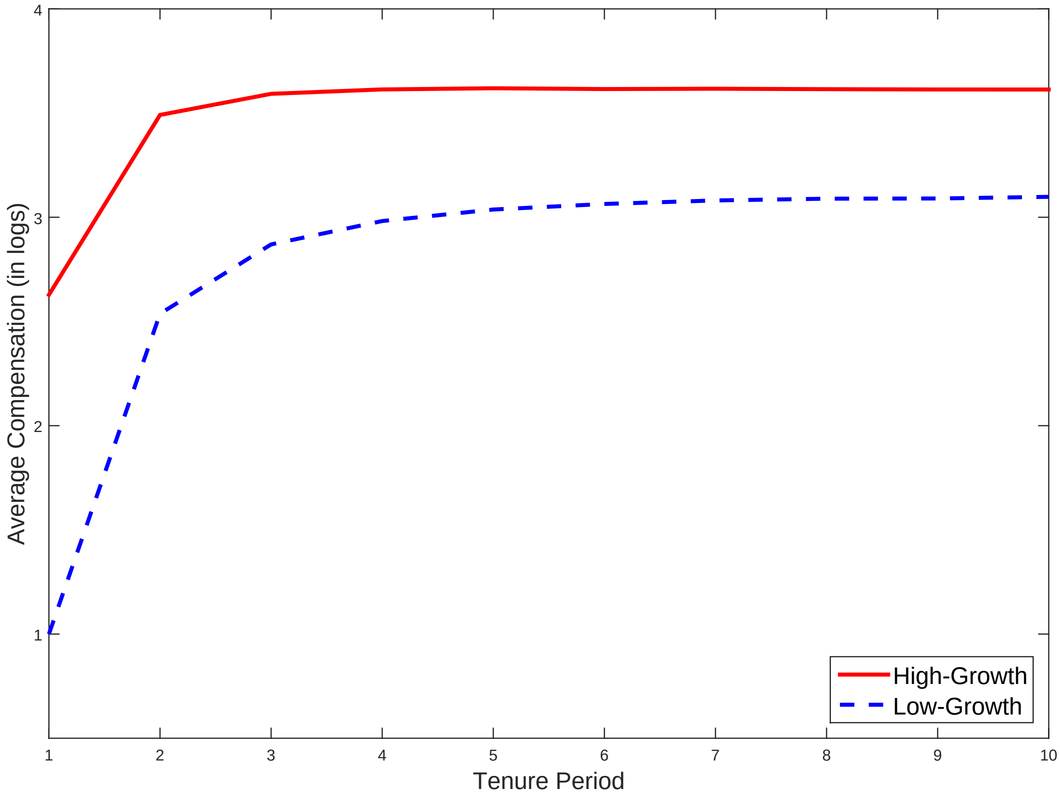

Figure 5: Compensation Profile over Tenure: High-Growth vs. Low-Growth

Notes: The figure depicts model-implied expected compensation, expressed in logs, as a function of time since hiring, conditional on being retained. Simulations are run for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\gamma = 0.25$ (high-growth) or $\gamma = 0.10$ (low-growth), $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. The size of the firm, identical in the two cases, is normalized so that the profile starts at 1 in the low-growth case. Optimal contractual thresholds are $w_{\mathrm{c}} = 1.24$ in the high-growth case, and $w_{\mathrm{g}} = 0.92$ and $w_{\mathrm{c}} = 1.36$ in the low-growth case. A period corresponds to a quarter.

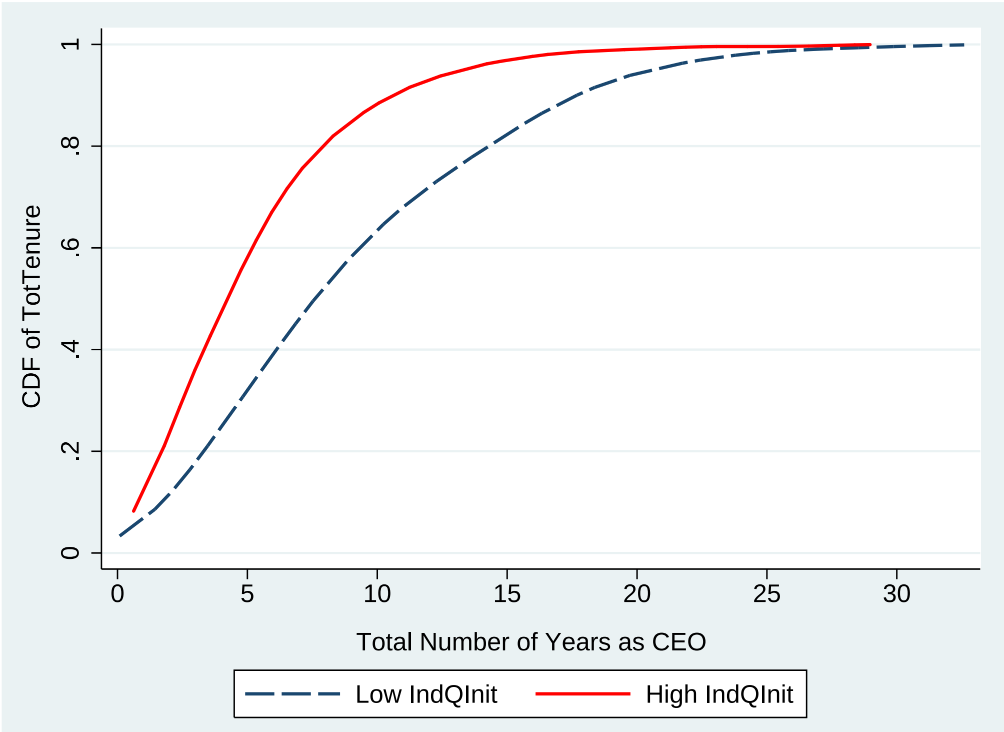

Figure 6: Distribution of CEO Tenure Length Conditional on Initial Growth Prospects

Notes: The figure depicts kernel estimates of the empirical cumulative distribution of CEO tenure length (*TotTenure*) for two sub-samples. The first sub-sample ('Low IndQInit') consists of the bottom quintile of CEO episodes sorted by initial industry Q; the corresponding distribution is plotted as a dashed line. The second sub-sample ('High IndQInit') consists of the top quintile of episodes sorted by initial industry Q; the corresponding distribution is plotted as a solid line. Details on variable definitions are provided in the main text and in Appendix G.

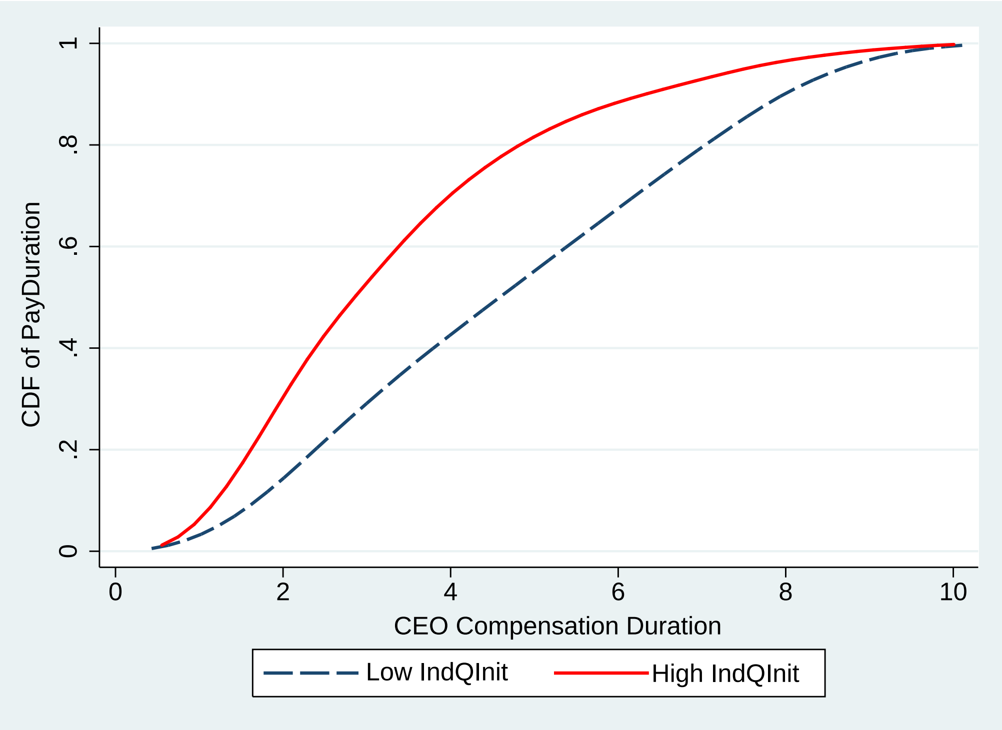

Figure 7: Distribution of CEO Pay Duration Conditional on Initial Growth Prospects

Notes: The figure depicts kernel estimates of the empirical cumulative distribution of realized compensation duration (*PayDuration*) for the bottom and top quintiles of CEO episodes sorted by initial industry Q (*IndQInit*), denoted by 'Low IndQInit' and 'High IndQInit', respectively. Details on variable definitions are provided in the main text and in Appendix G.

Table 1: Summary Statistics

Variable	Mean	Sd	p25	p50	p75	N
TotTenure	6.404	4.935	3.000	5.000	9.000	2,398
Turnover	0.084	0.277	0.000	0.000	0.000	26,957
LnTotPay	7.856	1.037	7.128	7.851	8.581	30,599
IndQInit	1.763	0.815	1.153	1.523	2.119	30,599
RatioQ	1.047	0.317	0.868	1.013	1.171	30,599
CAR	0.002	0.240	-0.141	-0.003	0.131	30,599
ROA	0.039	0.076	0.013	0.040	0.075	30,599
LnAssets	7.670	1.703	6.424	7.583	8.859	30,559

Notes: The table reports summary sample statistics for the merged ExecuComp/Compustat/CRSP data set, which covers CEO episodes reported in ExecuComp over the period 1992-2013. *TotTenure* is the total number of tenure years for CEO episodes that have finished within the sample period. *Turnover* is a dummy variable which equals 1 if the CEO is replaced in the following year and zero otherwise. *LnTotPay* is the logarithm of total CEO compensation awarded in a given calendar year. *IndQInit* is the arithmetic mean of the average Q of all firms in the same 4-digit SIC industry group in the year before the CEO was appointed; the same value is repeated throughout each CEO episode. *RatioQ* is the ratio of the mean industry average Q in a given year divided by *IndQInit*. *CAR* is the two-year cumulative abnormal return of the firm (annualized). *ROA* is return on assets. *LnAssets* is the logarithm of the book value of total assets. Further details on variable definitions are provided in the main text and in Appendix G.

Table 2: Determinants of CEO Turnover

	(A) Coefficients b/se	(B) Marginal Effects b/se	(C) Coefficients of Variation (in percentage points)
IndQInit	0.070***	0.010***	0.815
	(0.015)	(0.002)	
RatioQ	0.145***	0.021***	0.667
	(0.042)	(0.006)	
CAR	-0.653***	-0.093***	-2.232
	(0.054)	(0.008)	
ROA	-1.003***	-0.143***	-1.087
	(0.139)	(0.020)	
LnAssets	0.038***	0.005***	0.852
	(0.007)	(0.001)	
N	26,957	26,957	26,957

Notes: The table summarizes the evidence on the probability of CEO turnover from the probit regression estimated over the merged ExecuComp/Compustat/CRSP data set from 1992 to 2013. The dependent variable is the *Turnover* indicator variable. *IndQInit* is the arithmetic mean of the average Q of all firms in the same 4-digit SIC industry group in the year before the CEO was appointed; the same value is repeated throughout each CEO episode. *RatioQ* is the ratio of the mean industry average Q in a given year divided by *IndQInit*. *CAR* is the two-year cumulative abnormal return of the firm (annualized). *ROA* is return on assets. *LnAssets* is the logarithm of the book value of total assets. Calendar year fixed effects are included in the regression.

Table 3: Initial Growth Prospects and Growth-Induced Turnover

	Marginal Effect of *RatioQ* b/se
Low *IndQInit*	0.019***
	(0.005)
Median *IndQInit*	0.020***
	(0.006)
High *IndQInit*	0.022***
	(0.006)
N	26,957

Notes: The table reports the marginal effect of *RatioQ* on the likelihood of turnover for different levels of initial industry Q, as implied by the probit model estimated over the merged ExecuComp/Compustat/CRSP data set from 1992 to 2013. *IndQInit* is the arithmetic mean of the average Q of all firms in the same 4-digit SIC industry group in the year before the CEO was appointed; the same value is repeated throughout each CEO episode. *RatioQ* is the ratio of the mean industry average Q in a given year divided by *IndQInit*. The marginal effect of *RatioQ* is evaluated at different quantiles of the distribution of *IndQInit*. 'Low', 'Median', and 'High' quantiles correspond to the 20-th, 50-th, and 80-th percentiles of the distribution of *IndQInit*, respectively.

Table 4: Determinants of Compensation Duration

	(A) b/se	(B) b/se	(C) b/se	(D) b/se
IndQInit	-0.331***	-0.295***	-0.153**	-0.140**
	(0.057)	(0.057)	(0.064)	(0.065)
Year Fixed Effects	No	Yes	No	Yes
Industry Fixed Effects	No	No	Yes	Yes
R-squared	0.014	0.165	0.058	0.191
N	2,038	2,038	2,038	2,038

Notes: This table summarizes the evidence on CEO compensation duration. The regression is estimated over CEO episodes reported in ExecuComp over the period 1992-2013. The dependent variable is the measure of pay duration defined in Section 4.3. *IndQInit* is the arithmetic mean of the average Q of all firms in the same 4-digit SIC industry group in the year before the CEO was appointed. Industry fixed effects are based on 1-digit SIC codes.

Table 5: Determinants of CEO Compensation

	(A) *LnTotPay*	(B) *LnTotPay*
TenureYear	0.019***	0.009**
	(0.003)	(0.003)
IndQInit	0.114***	0.106***
	(0.016)	(0.017)
IndQInit $\times$ *TenureYear*	-0.012***	-0.009***
	(0.002)	(0.002)
CAR	0.556***	0.441***
	(0.022)	(0.017)
LnAssets	0.421***	0.226***
	(0.004)	(0.010)
Firm Fixed Effects	No	Yes
Industry Fixed Effects	Yes	No
Year Fixed Effects	Yes	Yes
R-squared	0.523	0.665
N	28,925	28,925

Notes: This table summarizes the evidence on the profile of CEO compensation over tenure. *LnTotPay* is the logarithm of total CEO pay awarded in a given year as reported in ExecuComp. *TenureYear* is the number of years in tenure of the CEO in a given calendar year. *IndQInit* is the arithmetic mean of the average Q of all firms in the same 4-digit SIC industry group in the year before the CEO was appointed. *CAR* is the two-year cumulative abnormal return of the firm (annualized). *ROA* is return on assets. *LnAssets* is the logarithm of the book value of total assets. The regression is estimated over all episode-year observations in our sample, some of which pertain to CEO episodes that have not finished by the end of the sample period.

A The Setting

In this appendix, we provide a complete description of the environment that we consider in this paper, in which an infinitely-lived firm is run by a sequence of managers who can divert cashflows for their own benefit. We build the model that we study on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a sequence of independent standard one-dimensional Brownian motions $Z^1, Z^2, \ldots, Z^n, \ldots$ as well as an independent sequence of independent and identically distributed random variables $U^1, U^2, \ldots, U^n, \ldots$, each having the uniform distribution on $[0, 1]$. We denote by $(\mathcal{F}_t^{Z^n})$ the natural filtration of Z^n. We assume that these filtrations as well as any other one we consider in this and the following appendices have been regularised to satisfy the "usual conditions", namely, to be right-continuous and augmented by the $\mathbb{P}$-negligible sets in $\mathcal{F}$.

A.1 The n-th Manager's Contract

In this section, we describe the contract of the n-th manager. To simplify the notation, we let $t = 0$ refer to the time at which the n-th manager takes office. We model the n-th manager's size-adjusted cumulative stealing strategy by an increasing continuous $(\mathcal{F}_t^{Z^n})$-adapted process A^n such that $A_0^n = 0$. We denote by $\mathcal{A}^n$ the family of all such processes. Given a stealing strategy $A^n \in \mathcal{A}^n$, we denote by $(\hat{\mathcal{F}}_t^n) = \big(\hat{\mathcal{F}}_t^n(A)\big)$ the information flow generated by the size-adjusted reported cashflows during the tenure of the n-th manager, which is the natural filtration of the process $\hat{Y}^n$ defined by

$$\hat{Y}_t^n = \mu t - A_t^n + \sigma Z_t^n.$$

It is worth noting that $\hat{\mathcal{F}}_t^n \subseteq \mathcal{F}_t^{Z^n}$ for all $t \geq 0$, with equality holding if $A^n = 0$.

We assume that the firm's growth policy is based on the history of reported cashflows during the tenure of each manager. Accordingly, we model the firm's growth policy during the tenure of the n-th manager by a càdlàg $(\hat{\mathcal{F}}_t^n)$-progressively measurable process process G^n with values in the interval $[0, 1]$. We define the time τ_{g}^n that elapses between the appointment of the n-th manager and his growth-induced dismissal (if he is not fired for disciplinary reasons before) by

$$\tau_{\mathrm{g}}^n = \inf \left\{ t \geq 0 \;\middle|\; \exp\left(-q \int_0^t G_s^n \, ds\right) \leq U^n \right\}, \tag{A.1}$$

with the usual convention that $\inf \emptyset = \infty$. In view of the independence of $(\hat{\mathcal{F}}_t^n)$ and U^n, we can see that

$$\mathbb{P}\left(\tau_{\mathrm{g}}^n > t \;\middle|\; \hat{\mathcal{F}}_t^n\right) = \mathbb{P}\left(U^n < \exp\left(-q \int_0^t G_s^n \, ds\right) \;\middle|\; \hat{\mathcal{F}}_t^n\right) = \exp\left(-q \int_0^t G_s^n \, ds\right).$$

We assume that disciplinary dismissal is also based on the history of each manager's reported cashflows. Accordingly, the time τ_{d}^n that elapses between the appointment of the n-th manager and his disciplinary firing (if he is not replaced for the sake of taking a growth opportunity before) is an $(\hat{\mathcal{F}}_t^n)$-stopping time. Furthermore, we assume that the manager's compensation is also determined based on the history of reported cashflows. Accordingly, the manager's size-adjusted cumulative compensation is given by an increasing càdlàg $(\hat{\mathcal{F}}_t^n)$-adapted process C^n such that $C_{0-} = 0$, while the manager's size-adjusted severance upon growth-induced dismissal is $S^n_{\tau^n_{\mathrm{g}}}$, where S^n is a positive $(\hat{\mathcal{F}}_t^n)$-progressively measurable process.

Remark A-1. If the firm stands ready to take all growth opportunities, setting $G^n \equiv 1$ for all n, then the times τ^n_{g} are independent and exponentially distributed with parameter q. Indeed, the choice $G^n \equiv 1$ gives rise to the identities

$$\mathbb{P}\left(\tau^n_{\mathrm{g}} > t \;\middle|\; \hat{\mathcal{F}}_t^n\right) = e^{-qt} = \mathbb{P}(\tau^n_{\mathrm{g}} > t).$$

In other words, the random times between the arrival of two consecutive growth opportunities are independent random variables that are exponentially distributed with parameter q. ∎

For technical reasons, we assume that the family $\mathcal{A}^n$ is restricted to include only processes satisfying the integrability condition

$$\mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t}\, dA_t^n\right] < \infty. \tag{A.2}$$

Given any such stealing strategy $A^n \in \mathcal{A}^n$, we use the following notation:

$\mathcal{P}^n_C(A^n)$ is the family of all increasing càdlàg $(\hat{\mathcal{F}}_t^n)$-adapted processes C^n such that $C^n_{0-} = 0$;

$\mathcal{P}^n_S(A^n)$ is the family of all positive càdlàg $(\hat{\mathcal{F}}_t^n)$-adapted processes S^n;

$\mathcal{P}^n_G(A^n)$ is the family of all $(\hat{\mathcal{F}}_t^n)$-progressively measurable processes G^n with values in $[0, 1]$;

$\mathcal{P}^n_{\tau_{\mathrm{d}}}(A^n)$ is the set of all $(\hat{\mathcal{F}}_t^n)$-stopping times.

These families depend on the choice of $A^n \in \mathcal{A}^n$ through the dependence on A^n of the natural filtration $(\hat{\mathcal{F}}_t^n)$ of the reported cashflows process $\hat{Y}^n$. In particular, it is worth noting that

$$\begin{gathered}\mathcal{P}^n_C(A^n) \subseteq \mathcal{P}^n_C(0), \quad \mathcal{P}^n_S(A^n) \subseteq \mathcal{P}^n_S(0), \\ \mathcal{P}^n_G(A^n) \subseteq \mathcal{P}^n_G(0) \quad \text{and} \quad \mathcal{P}^n_{\tau_{\mathrm{d}}}(A^n) \subseteq \mathcal{P}^n_{\tau_{\mathrm{d}}}(0) \quad \text{for all } A^n \in \mathcal{A}^n\end{gathered} \tag{A.3}$$

because $\hat{\mathcal{F}}_t^n \subseteq \mathcal{F}_t^{Z^n}$ for all $t \geq 0$, with equality if $A^n = 0$. Furthermore, we assume that the families $\mathcal{P}^n_C(A^n)$ and $\mathcal{P}^n_S(A^n)$ are restricted to include only processes satisfying the integrability condition

$$\mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t}\, dC_t^n + \sup_{t\geq 0}\bigl(e^{-\varrho t} S_t^n\bigr)\right] < \infty. \tag{A.4}$$

We can now introduce the formal definition of the n-th manager's contract.

Definition A-1. A *long-term incentive contract*, or just *incentive contract*, for the n-th manager is a function

$$\Gamma^n = (\Gamma^n_C, \Gamma^n_S, \Gamma^n_G, \Gamma^n_{\tau_\mathrm{d}}) : \mathcal{A}^n \to \mathcal{P}^n_C(0) \times \mathcal{P}^n_S(0) \times \mathcal{P}^n_G(0) \times \mathcal{P}^n_{\tau_\mathrm{d}}(0)$$

such that

$$\Gamma^n_C(A^n) \in \mathcal{P}^n_C(A^n), \quad \Gamma^n_S(A^n) \in \mathcal{P}^n_S(A^n),$$
$$\Gamma^n_G(A^n) \in \mathcal{P}^n_G(A^n) \quad \text{and} \quad \Gamma_{\tau_\mathrm{d}}(A^n) \in \mathcal{P}^n_{\tau_\mathrm{d}}(A^n) \quad \text{for all } A^n \in \mathcal{A}^n.$$

We denote by $\mathcal{G}^n$ the family of all such contracts. ∎

Remark A-2. It would be natural to include additional requirements as part of the definition of a long-term incentive contract. For instance, for any two stealing strategies that coincide up to a certain stopping time, the evaluation of a contract at these two strategies should result in the same compensation and termination outcomes up to that stopping time. We have opted for not spelling out explicitly such constraints in Definition A-1 because they do not affect the remainder of our analysis. ∎

A.2 The Managers' and the Firm's Payoffs

We define

$$\tau^1_\mathrm{h} = 0, \quad \tau^{n+1}_\mathrm{h} = \sum_{j=1}^n \tau^j_\mathrm{d} \wedge \tau^j_\mathrm{g}, \quad \Phi^1 = 1 \quad \text{and} \quad \Phi^{n+1} = (1+\gamma)^{\mathbf{1}_{\{\tau^1_\mathrm{g} \le \tau^1_\mathrm{d}\}} + \cdots + \mathbf{1}_{\{\tau^n_\mathrm{g} \le \tau^n_\mathrm{d}\}}}, \qquad \text{(A.5)}$$

for $n \geq 1$, and we note that τ^n_h is the time at which the n-th manager is hired, while Φ^n is the size of the firm during the n-th manager's tenure. Accordingly, $\Phi^n A^n$, $\Phi^n C^n$ and $\Phi^n S^n$ model the n-th manager's actual stealing strategy, cumulative compensation and severance upon dismissal, respectively. We also consider the σ-algebras

$$\mathcal{I}^1 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{I}^{n+1} = \sigma\left(\hat{Y}^j_{t \wedge \tau^j_\mathrm{d} \wedge \tau^j_\mathrm{g}}, \, \tau^j_\mathrm{d}, \, \tau^j_\mathrm{g}, \ j = 1, \ldots, n, \ t \geq 0\right), \qquad \text{(A.6)}$$

for $n \geq 1$, and we note that $\mathcal{I}^n$ is the information that is available to the firm at the hiring time τ^n_h of the n-th manager. In view of the independence of (Z^1, U^1), …, (Z^n, U^n), … and the structure of each manager's contract that we considered in the previous section, we can see that

$$\tau^n_\mathrm{d}, \ \tau^n_\mathrm{g}, \ \Phi^{n+1} \text{ are } \mathcal{I}^{n+1}\text{-measurable}, \qquad \text{(A.7)}$$

while

$$A^j, \ S^j, \ C^j, \ \tau^j_\mathrm{d}, \ \tau^j_\mathrm{g}, \ \frac{\Phi^{j+1}}{\Phi^n}, \ \text{for } j \geq n, \ \text{are independent of } \mathcal{I}^n. \qquad \text{(A.8)}$$

Given an incentive contract $\Gamma^n \in \mathcal{G}^n$ (see Definition A-1), the n-th manager's total expected discounted payoff as of the time τ_{h}^n of his hiring is given by

$$\tilde{M}^n(\Gamma^n, A^n \mid \mathcal{I}^n) = \mathbb{E}\bigg[\int_{[0,\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n[} e^{-\varrho t}\Phi^n \, dC_t^n + e^{-\varrho \tau_{\mathrm{d}}^n}\Phi^n \Delta C_{\tau_{\mathrm{d}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{d}}^n < \tau_{\mathrm{g}}^n\}} \\ + e^{-\varrho \tau_{\mathrm{g}}^n}\Phi^n S_{\tau_{\mathrm{g}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{g}}^n \le \tau_{\mathrm{d}}^n\} \cap \{\tau_{\mathrm{g}}^n < \infty\}} + \lambda \int_0^{\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n} e^{-\varrho t}\Phi^n \, dA_t^n \;\bigg|\; \mathcal{I}^n\bigg].$$

Here,

$$\text{we write } C^n,\ S^n \text{ and } \tau_{\mathrm{d}}^n \text{ in place of } \Gamma_C^n(A^n),\ \Gamma_S^n(A^n) \text{ and } \Gamma_{\tau_{\mathrm{d}}}^n(A^n),$$

respectively, and we note that

$$\tau_{\mathrm{g}}^n \text{ is defined as in (A.1) for } G^n = \Gamma_G^n(A^n).$$

In view of (A.7)–(A.8), we can see that this expression is equivalent to

$$\begin{aligned}\tilde{M}^n(\Gamma^n, A^n \mid \mathcal{I}^n) &= \Phi^n \, \mathbb{E}\bigg[\int_{[0,\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n[} e^{-\varrho t} \, dC_t^n + e^{-\varrho \tau_{\mathrm{d}}^n} \Delta C_{\tau_{\mathrm{d}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{d}}^n < \tau_{\mathrm{g}}^n\}} \\ &\qquad + e^{-\varrho \tau_{\mathrm{g}}^n} S_{\tau_{\mathrm{g}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{g}}^n \le \tau_{\mathrm{d}}^n\} \cap \{\tau_{\mathrm{g}}^n < \infty\}} + \lambda \int_0^{\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n} e^{-\varrho t} \, dA_t^n\bigg] \\ &=: \Phi^n M(\Gamma^n, A^n), \end{aligned} \tag{A.9}$$

where $M(\Gamma^n, A^n)$ is the n-th manager's size-adjusted total expected discounted payoff as of time τ_{h}^n.

From this point onward, we restrict our attention to admissible contracts. These contracts are such that they make no stealing optimal for the manager, i.e., no stealing is "incentive compatible". Additionally, these contracts are such that the manager's size-adjusted expected discounted compensation under no stealing is equal to $\bar{w}$.

Definition A-2. An *admissible long-term incentive contract*, or just *admissible contract*, for the n-th manager is any incentive contract $\Gamma^n \in \mathcal{G}^n$ (see Definition A-1) satisfying the admissibility constraints

$$M(\Gamma^n, 0) = \sup_{A^n \in \mathcal{A}^n} M(\Gamma^n, A^n) \quad \text{and} \quad M(\Gamma^n, 0) = \bar{w}. \tag{A.10}$$

We denote by $\mathcal{G}_{\mathrm{a}}^n \subseteq \mathcal{G}^n$ the family of all such contracts. ∎

We henceforth assume that the firm offers admissible contracts to all managers, who all refrain from stealing, namely, $A^n = 0$ for all $n \ge 1$. The expected discounted profit as of time τ_{h}^n that the firm receives during the tenure of the n-th manager is

$$\begin{aligned}\tilde{\Pi}^n(\Gamma^n \mid \mathcal{I}^n) &= \Phi^n \, \mathbb{E}\bigg[\int_0^{\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n} e^{-rt}\mu \, dt - \int_{[0,\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n[} e^{-rt} \, dC_t^n \\ &\qquad - e^{-r\tau_{\mathrm{d}}^n} \Delta C_{\tau_{\mathrm{d}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{d}}^n < \tau_{\mathrm{g}}^n\}} - e^{-r\tau_{\mathrm{g}}^n} S_{\tau_{\mathrm{g}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{g}}^n \le \tau_{\mathrm{d}}^n\} \cap \{\tau_{\mathrm{g}}^n < \infty\}}\bigg] \\ &=: \Phi^n \Pi(\Gamma^n), \end{aligned} \tag{A.11}$$

where $\Pi(\Gamma^n)$ is the size-adjusted expected discounted profit as of time $\tau_{\rm h}^n$ that the firm receives during the tenure of the n-th manager. Here, as well as in what follows,

$$\text{we write } C^n,\ S^n \text{ and } \tau_{\rm d}^n \text{ in place of } \Gamma_C^n(0),\ \Gamma_S^n(0) \text{ and } \Gamma_{\tau_{\rm d}}^n(0),$$

respectively, and we note that

$$\tau_{\rm g}^n \text{ is defined as in (A.1) for } G^n = \Gamma_G^n(0).$$

In view of (A.7)–(A.8), we can see that the expected discounted profit of the firm at the start of the n-th manager's tenure is

$$\begin{aligned}
\tilde{F}^n\left((\Gamma^j)_{j\geq n} \mid \mathcal{I}^n\right) &= \mathbb{E}\left[\Phi^n\Pi\big(\Gamma^n\big) + \sum_{j=n+1}^{\infty} e^{-r(\tau_{\rm h}^j-\tau_{\rm h}^n)}\big(\Phi^j\Pi(\Gamma^j) - \Phi^{j-1}\kappa\big) \;\Big|\; \mathcal{I}^n\right] \\
&= \Phi^n\left(\Pi(\Gamma^n) + \mathbb{E}\left[\sum_{j=n+1}^{\infty} e^{-r(\tau_{\rm h}^j-\tau_{\rm h}^n)}\left(\frac{\Phi^j}{\Phi^n}\Pi(\Gamma^j) - \frac{\Phi^{j-1}}{\Phi^n}\kappa\right)\right]\right) \\
&=: \Phi^n F^n\left((\Gamma^j)_{j\geq n}\right),
\end{aligned}$$

where $F^n\left((\Gamma^j)_{j\geq n}\right)$ is the size-adjusted expected discounted profit of the firm at the start of the n-th manager's tenure. Note that this expression incorporates the turnover costs associated with the n-th manager and his successors. In view of the identities

$$\begin{aligned}
\tau_{\rm h}^j - \tau_{\rm h}^n &= \tau_{\rm d}^n \wedge \tau_{\rm g}^n + \cdots + \tau_{\rm d}^{j-1} \wedge \tau_{\rm g}^{j-1} \\
&= \tau_{\rm d}^n \wedge \tau_{\rm g}^n + \tau_{\rm h}^j - \tau_{\rm h}^{n+1} \quad \text{for all } j \geq n+1,
\end{aligned}$$

which follow from (A.5), we can see that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=n+1}^{\infty} e^{-r(\tau_{\rm h}^j-\tau_{\rm h}^n)}\left(\frac{\Phi^j}{\Phi^n}\Pi(\Gamma^j) - \frac{\Phi^{j-1}}{\Phi^n}\kappa\right)\right] \\
&\quad = \mathbb{E}\left[e^{-r(\tau_{\rm d}^n\wedge\tau_{\rm g}^n)}\left\{\frac{\Phi^{n+1}}{\Phi^n}\Pi(\Gamma^{n+1}) + \sum_{j=n+2}^{\infty} e^{-r(\tau_{\rm h}^j-\tau_{\rm h}^{n+1})}\left(\frac{\Phi^j}{\Phi^n}\Pi(\Gamma^j) - \frac{\Phi^{j-1}}{\Phi^n}\kappa\right) - \kappa\right\}\right] \\
&\quad = \mathbb{E}\left[e^{-r(\tau_{\rm d}^n\wedge\tau_{\rm g}^n)}\frac{1}{\Phi^n}\left\{\mathbb{E}\left[\Phi^{n+1}\Pi(\Gamma^{n+1})\right.\right.\right. \\
&\qquad \left.\left.\left. + \sum_{j=n+2}^{\infty} e^{-r(\tau_{\rm h}^j-\tau_{\rm h}^{n+1})}\left(\Phi^j\Pi(\Gamma^j) - \Phi^{j-1}\kappa\right) \;\Big|\; \mathcal{I}^{n+1}\right] - \Phi^n\kappa\right\}\right] \\
&\quad = \mathbb{E}\left[e^{-r(\tau_{\rm d}^n\wedge\tau_{\rm g}^n)}\frac{\Phi^{n+1}}{\Phi^n}F^{n+1}\left((\Gamma^j)_{j\geq n+1}\right) - e^{-r(\tau_{\rm d}^n\wedge\tau_{\rm g}^n)}\kappa\right].
\end{aligned}$$

We thus obtain the recursive expression

$$F^n\left((\Gamma^j)_{j\geq n}\right) = \Pi(\Gamma^n) + \mathbb{E}\left[e^{-r(\tau_{\rm d}^n\wedge\tau_{\rm g}^n)}\frac{\Phi^{n+1}}{\Phi^n}F^{n+1}\left((\Gamma^j)_{j\geq n+1}\right) - e^{-r(\tau_{\rm d}^n\wedge\tau_{\rm g}^n)}\kappa\right].$$

Combining this result with the calculation

$$\mathbb{E}\left[e^{-r(\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n)} \frac{\Phi^{n+1}}{\Phi^n}\right] = \mathbb{E}\left[e^{-r(\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n)} (1+\gamma)^{\mathbf{1}_{\{\tau_{\mathrm{g}}^n \le \tau_{\mathrm{d}}^n\}}}\right]$$
$$= \mathbb{E}\left[e^{-r\tau_{\mathrm{d}}^n} \mathbf{1}_{\{\tau_{\mathrm{d}}^n < \tau_{\mathrm{g}}^n\}} + (1+\gamma) e^{-r\tau_{\mathrm{g}}^n} \mathbf{1}_{\{\tau_{\mathrm{g}}^n \le \tau_{\mathrm{d}}^n\}}\right]$$

and (A.11), we can see that the size-adjusted expected discounted profit of the firm at the hiring time τ_{h}^n of the n-th manager satisfies the recursive equation

$$\begin{aligned}
&F^n\left((\Gamma^j)_{j\ge n}\right) \\
&= \mathbb{E}\Bigg[\int_0^{\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n} e^{-rt} \mu \, dt - \int_{[0, \tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n[} e^{-rt} \, dC_t^n \\
&\quad + e^{-r\tau_{\mathrm{d}}^n} \left[F^{n+1}\left((\Gamma^j)_{j\ge n+1}\right) - \Delta C_{\tau_{\mathrm{d}}^n}^n - \kappa\right] \mathbf{1}_{\{\tau_{\mathrm{d}}^n < \tau_{\mathrm{g}}^n\}} \\
&\quad + e^{-r\tau_{\mathrm{g}}^n} \left[(1+\gamma) F^{n+1}\left((\Gamma^j)_{j\ge n+1}\right) - S_{\tau_{\mathrm{g}}^n}^n - \kappa\right] \mathbf{1}_{\{\tau_{\mathrm{g}}^n \le \tau_{\mathrm{d}}^n\} \cap \{\tau_{\mathrm{g}}^n < \infty\}}\Bigg]. \qquad \text{(A.12)}
\end{aligned}$$

A.3 The Firm's Problem: Take One

The uncertainty under the n-th manager's tenure is driven by (Z^n, U^n), which is an independent copy of (Z^1, U^1). In particular, $(\mathcal{A}^n)$ is a sequence of independent copies of $\mathcal{A}^1$. As a result, if a contract is admissible (in the sense of Definition A-2) for the first manager, then the same contract is admissible for all managers. In view of this observation and a simple induction argument along the lines that lead to the recursive equation (A.12), we can see that it is optimal for the firm seeking to implement no stealing to give the same admissible contract to all managers.

If all successive managers are offered the same contract $\Gamma \in \mathcal{G}_{\mathrm{a}}$, then the firm's size-adjusted expected discounted profit at the start of any manager's tenure is

$$F^1\big((\Gamma, \ldots, \Gamma, \ldots)\big) = F^n\big((\Gamma, \ldots, \Gamma, \ldots)\big) =: F(\Gamma) \quad \text{for all } n \ge 1.$$

In particular, (A.12) implies that

$$\begin{aligned}
F(\Gamma) = \mathbb{E}\Bigg[\int_0^{\tau_{\mathrm{d}} \wedge \tau_{\mathrm{g}}} e^{-rt} \mu \, dt - \int_{[0, \tau_{\mathrm{d}} \wedge \tau_{\mathrm{g}}[} e^{-rt} \, dC_t + e^{-r\tau_{\mathrm{d}}} \Big[F(\Gamma) - \Delta C_{\tau_{\mathrm{d}}} - \kappa\Big] \mathbf{1}_{\{\tau_{\mathrm{d}} < \tau_{\mathrm{g}}\}} \\
+ e^{-r\tau_{\mathrm{g}}} \Big[(1+\gamma) F(\Gamma) - S_{\tau_{\mathrm{g}}} - \kappa\Big] \mathbf{1}_{\{\tau_{\mathrm{g}} \le \tau_{\mathrm{d}}\} \cap \{\tau_{\mathrm{g}} < \infty\}}\Bigg], \qquad \text{(A.13)}
\end{aligned}$$

where C, S, and τ_{d} stand for $\Gamma_C(0)$, $\Gamma_S(0)$ and $\Gamma_{\tau_{\mathrm{d}}}(0)$, respectively, and τ_{g} is defined as in (A.1) for $G = \Gamma_G(0)$.

We conclude with the statement of the contracting problem that the firm faces.

Problem A-1. Determine an admissible contract $\Gamma^\star \in \mathcal{G}_{\mathrm{a}}$ (see Definition A-2) such that

$$F(\Gamma^\star) = \sup_{\Gamma \in \mathcal{G}_{\mathrm{a}}} F(\Gamma).$$

■

B Admissible Dynamic Contracts

In light of the analysis in Appendix A, we now focus on the first manager's contract. To simplify the notation, we write Z, $(\mathcal{F}_t^Z)$, $\mathcal{A}$, $(\hat{\mathcal{F}}_t) = \big(\hat{\mathcal{F}}_t(A)\big)$, etc, instead of Z^1, $(\mathcal{F}_t^{Z^1})$, $\mathcal{A}^n$, $(\hat{\mathcal{F}}_t^n) = \big(\hat{\mathcal{F}}_t^n(A)\big)$, etc, in what follows. In particular, given a stealing strategy $A \in \mathcal{A}$,

$$\begin{aligned}
&\mathcal{P}_C(A) \text{ is the family of all increasing càdlàg } (\hat{\mathcal{F}}_t)\text{-adapted process } C \\
&\qquad \text{such that } C_{0-} = 0; && \text{(B.1)}\\
&\mathcal{P}_S(A) \text{ is the family of all positive càdlàg } (\hat{\mathcal{F}}_t)\text{-adapted processes } S; && \text{(B.2)}\\
&\mathcal{P}_G(A) \text{ is the family of all } (\hat{\mathcal{F}}_t)\text{-progressively measurable processes } G \\
&\qquad \text{with values in } [0,1]; && \text{(B.3)}\\
&\mathcal{P}_{\tau_{\rm d}}(A) \text{ is the set of all } (\hat{\mathcal{F}}_t)\text{-stopping times} && \text{(B.4)}
\end{aligned}$$

(recall that the families $\mathcal{P}_C(A)$ and $\mathcal{P}_S(A)$ are restricted to include only processes satisfying the integrability condition (A.4)). We will also need the filtration $(\mathcal{F}_t^{Z,\tau_{\rm g}})$ that is larger than $(\mathcal{F}_t^Z)$ and incorporates the information on the occurrence of growth-induced termination, which is defined by

$$\mathcal{F}_t^{Z,\tau_{\rm g}} = \mathcal{F}_t^Z \vee \sigma\Big(\{\tau_g \leq s\}\,,\, s \leq t\Big)$$

(see also the discussion on filtrations at the very beginning of Appendix A). Notice that if the manager refrains from stealing ($A = 0$), then $\hat{\mathcal{F}}_t = \mathcal{F}_t^Z$ for all $t \geq 0$, in which case the sigma-algebra $\mathcal{F}_t$ introduced in Section 1.2 coincides with $\mathcal{F}_t^{Z,\tau_{\rm g}}$.

B.1 Proof of Lemma 1

Lemma 1 is a direct consequence of the following result (see Remark B-1 below).

Lemma B-1. *Consider any processes $C \in \mathcal{P}_C(0)$, $S \in \mathcal{P}_S(0)$, $G \in \mathcal{P}_G(0)$ together with any stopping time $\tau_{\rm d} \in \mathcal{P}_{\tau_{\rm d}}(0)$, and let $\tau_{\rm g}$ be the random time that is defined as in (A.1). Also, consider the processes M, $\tilde{M}$ defined by*

$$\begin{aligned}
M_t = \mathbf{1}_{\{t<\tau_{\rm d}\wedge\tau_{\rm g}\}}\,\mathbb{E}\bigg[\int_{]t,\tau_{\rm d}\wedge\tau_{\rm g}[} e^{-\varrho(s-t)}\,dC_s \\
+ e^{-\varrho(\tau_{\rm d}-t)}\Delta C_{\tau_{\rm d}}\mathbf{1}_{\{\tau_{\rm d}<\tau_{\rm g}\}} + e^{-\varrho(\tau_{\rm g}-t)} S_{\tau_{\rm g}}\mathbf{1}_{\{\tau_{\rm g}\leq\tau_{\rm d}\}\cap\{\tau_{\rm g}<\infty\}} \;\Big|\; \mathcal{F}_t^{Z,\tau_{\rm g}}\bigg],
\end{aligned}$$

$$\tilde{M}_t = \mathbf{1}_{\{t<\tau_{\rm d}\}}\mathfrak{d}_t^{-1}\,\mathbb{E}\left[\int_{]t,\tau_{\rm d}]\cap\mathbb{R}_+} \mathfrak{d}_s\,dC_s + q\int_t^{\tau_{\rm d}} \mathfrak{d}_s G_s S_s\,ds \;\Big|\; \mathcal{F}_t^Z\right],$$

where

$$\mathfrak{d}_t = \exp\left(-\varrho t - \int_0^t qG_s\,ds\right), \tag{B.5}$$

and let $T^0_{\tilde M}$ be the first hitting time of 0 by $\tilde M$, namely,

$$T^0_{\tilde M} = \inf\{t \geq 0 \mid \ \tilde M_t = 0\}.$$

The following statements hold true:

(I) *$M_t = \mathbf{1}_{\{t<\tau_{\mathrm{g}}\}} \tilde M_t$ for all $t \geq 0$.*

(II) *There exists an $(\mathcal{F}^Z_t)$-progressively measurable process β such that $\int_0^T \mathfrak{d}_t^2 \beta_t^2 \, dt < \infty$ for all $T > 0$, and*

$$dM_t = \big[\varrho M_t + qG_t(M_t - S_t)\big] \, dt - dC_t + \sigma \beta_t \, dZ_t \tag{B.6}$$

on the event $\{t < \tau_{\mathrm{d}} \wedge \tau_{\mathrm{g}}\}$.

(III) *$T^0_{\tilde M} \leq \tau_{\mathrm{d}}$, and the processes $(C - C_{T^0_{\tilde M}})\mathbf{1}_{]T^0_{\tilde M}, \tau_{\mathrm{d}}]}$, $GS\mathbf{1}_{]T^0_{\tilde M}, \tau_{\mathrm{d}}]}$ and $\beta \mathbf{1}_{]T^0_{\tilde M}, \tau_{\mathrm{d}}]}$ are indistinguishable from 0.*

Proof. Claim (I) follows from standard credit risk theory (e.g., see Bielecki and Rutkowski (2002), Section 5.1.1). To establish the rest of the claims, we first observe that

$$\begin{aligned} \mathbb{E}\bigg[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_s \, dC_s + q \int_0^{\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds \ \Big| \ \mathcal{F}^Z_t \bigg] & \\ = \int_{[0,t\wedge\tau_{\mathrm{d}}]} \mathfrak{d}_s \, dC_s + q \int_0^{t\wedge\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds & \\ + \mathbf{1}_{\{t<\tau_{\mathrm{d}}\}} \mathbb{E}\left[\int_{]t,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_s \, dC_s + q \int_t^{\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds \ \Big| \ \mathcal{F}^Z_t \right] & \\ = \int_{[0,t\wedge\tau_{\mathrm{d}}]} \mathfrak{d}_s \, dC_s + q \int_0^{t\wedge\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds + \mathfrak{d}_t \tilde M_t. & \end{aligned}$$

In view of the martingale representation theorem, there exists an $(\mathcal{F}^Z_t)$-progressively measurable process β such that $\int_0^T \mathfrak{d}_s^2 \beta_s^2 \, ds < \infty$ for all $T > 0$, and

$$\mathbb{E}\left[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_s \, dC_s + q \int_0^{\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds \ \Big| \ \mathcal{F}^Z_t \right] = \tilde M_{0-} + \int_0^t \mathfrak{d}_s \beta_s \, dZ_s,$$

where

$$\tilde M_{0-} = \mathbb{E}\left[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_s \, dC_s + q \int_0^{\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds \right] = \Delta C_0 + \tilde M_0.$$

Rearranging terms, we obtain

$$\mathfrak{d}_t \tilde M_t = \tilde M_{0-} - \int_{[0,t\wedge\tau_{\mathrm{d}}]} \mathfrak{d}_s \, dC_s - q \int_0^{t\wedge\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds + \int_0^t \mathfrak{d}_s \beta_s \, dZ_s. \tag{B.7}$$

Using the definition of $\mathfrak{d}$ in (B.5), (B.7), and the integration by parts formula, we can see that $\tilde M$ satisfies

$$d\tilde M_t = \big[\varrho \tilde M_t + qG_t(\tilde M_t - S_t)\big] \, dt - dC_t + \sigma \beta_t \, dZ_t$$

on the event $\{t < \tau_{\rm d}\}$, which, combined with part (I) of the lemma, implies the claim in (II). Furthermore, the definitions of $\tilde{M}$ and $T^0_{\tilde{M}}$, along with (B.7), imply all the properties listed in (III). ∎

Remark B-1. It is straightforward to see that Lemma 1 follows immediately from Lemma B-1. Indeed, consider a long-term incentive contract $\Gamma \in \mathcal{G}$ and suppose the manager refrains from stealing, so that $\mathcal{F}_t^{Z,\tau_{\rm g}}$ represents all the information accumulated by the firm up to time t. Also, let $C = \Gamma_C(0)$, $S = \Gamma_S(0)$ and $\tau_{\rm d} = \Gamma_{\tau_{\rm d}}(0)$, and let $\tau_{\rm g}$ be the random time that is defined as in (A.1) for $G = \Gamma_G(0)$. In view of Definition A-1, $C \in \mathcal{P}_C(0)$, $S \in \mathcal{P}_S(0)$, $G \in \mathcal{P}_G(0)$ and $\tau_{\rm d} \in \mathcal{P}_{\tau_{\rm d}}(0)$. Therefore, the results of Lemma B-1 apply, and Lemma 1 follows from Claim (II). ∎

B.2 Dynamic Contracts

Under no stealing, the dynamics given by (B.6) identify with

$$dM_t = \big[\varrho M_t + qG_t(M_t - S_t)\big]\, dt - dC_t + \beta_t\, (d\hat{Y}_t - \mu\, dt)$$

because, in this case, $\sigma\, dZ_t = d\hat{Y}_t - \mu\, dt$. This observation motivates us to restrict our attention to "dynamic" contracts that track the state process W whose stochastic dynamics are modelled by

$$\begin{aligned} dW_t &= \big[\varrho W_t + qG_t(W_t - S_t)\big]\, dt - dC_t + \beta_t\, (d\hat{Y}_t - \mu\, dt) \\ &= \big[\varrho W_t + qG_t(W_t - S_t)\big]\, dt - dC_t - \beta_t\, dA_t + \sigma\beta_t\, dZ_t, \end{aligned} \tag{B.8}$$

the second equality following from the fact that, in general, $d\hat{Y}_t - \mu\, dt = -dA_t + \sigma dZ_t$.

In view of these considerations, we adopt the following definition of dynamic contracts, where, in line with (B.1)–(B.4),

$$\begin{gathered} \mathcal{P}_\beta(A) \text{ is the family of all positive } (\hat{\mathcal{F}}_t)\text{-progressively measurable processes } \beta \\ \text{such that } \mathbb{E}\left[\int_0^\infty e^{-2rt}\beta_t^2\, dt\right] < \infty, \end{gathered} \tag{B.9}$$

where $r < \varrho$ is the firm's discount rate. Note that, for the purposes of our analysis below, we impose technical conditions on the process β that are stronger than the ones appearing in Lemma B-1.(II).

Definition B-1. A *dynamic contract* is a function

$$\overline{\mathrm{D}} = (\overline{\mathrm{D}}_C, \overline{\mathrm{D}}_S, \overline{\mathrm{D}}_G, \overline{\mathrm{D}}_{\tau_{\rm d}}, \overline{\mathrm{D}}_\beta) : \mathcal{A} \to \mathcal{P}_C(0) \times \mathcal{P}_S(0) \times \mathcal{P}_G(0) \times \mathcal{P}_{\tau_{\rm d}}(0) \times \mathcal{P}_\beta(0)$$

together with a constant $w_{\rm init} > 0$ such that

(I) $(\overline{\mathrm{D}}_C, \overline{\mathrm{D}}_S, \overline{\mathrm{D}}_G, \overline{\mathrm{D}}_{\tau_{\rm d}})$ is a contract in the sense of Definition A-1;

(II) $\overline{\mathrm{D}}_\beta(A) \in \mathcal{P}_\beta(A)$ for all $A \in \mathcal{A}$;

(III) given any $A \in \mathcal{A}$, the solution W to the SDE (B.8) for

$$C = \overline{\mathrm{D}}_C(A), \quad S = \overline{\mathrm{D}}_S(A), \quad G = \overline{\mathrm{D}}_G(A), \quad \beta = \overline{\mathrm{D}}_\beta(A) \quad \text{and} \quad W_{0-} = w_{\rm init},$$

is such that

(a) $\Delta C_t \leq W_{t-}$ for all $t \leq \tau_{\mathrm{d}} := \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}(A)$, and

(b) if $T_W^0 = \inf\{t \geq 0 \mid \; W_t = 0\}$, then $T_W^0 \leq \tau_{\mathrm{d}}$ and the processes $(C - C_{T_W^0})\mathbf{1}_{]T_W^0,\tau_{\mathrm{d}}]}$, $GS\mathbf{1}_{]T_W^0,\tau_{\mathrm{d}}]}$ and $\beta\mathbf{1}_{]T_W^0,\tau_{\mathrm{d}}]}$ are indistinguishable from 0.

We denote by $\overline{\mathcal{D}}^{w_{\mathrm{init}}}$ the family of all dynamic contracts that are associated with the initial condition $W_{0-} = w_{\mathrm{init}}$. ∎

Remark B-2. From a mathematical point of view, the *raison d'être* of the extra component that differentiates dynamic contracts from contracts in the sense of Definition A-1, namely, the process $\beta = \overline{\mathrm{D}}_\beta(A)$, is to give sense to the constraints in (III) that involve the solution W to the SDE (B.8). The constraints in (III).(b) are imposed in the definition of a dynamic contract to mimic the properties stated in Lemma B-1.(III). ∎

Remark B-3. The general definition of a dynamic contract that we have adopted here is slightly different from the one given in Section 1.4, in the sense that Definition B-1 allows for any random disciplinary dismissal time $\tau_{\mathrm{d}} \geq T_W^0$, whereas the exposition in the main text assumes that this random time identifies with the first hitting time of zero by W, namely, $\tau_{\mathrm{d}} = T_W^0$. The advantage of adopting a more general definition, as we do here, is that it allows us to derive the identity $\tau_{\mathrm{d}} = T_W^0$ as an incentive-compatibility requirement (see Lemma B-2). However, note that the notion of admissible dynamic contract that we define in the next section (see Definition B-3) coincides with the one introduced in Section 1.4. ∎

We can now introduce a general notion of admissibility for dynamic contracts, in line with Definition A-2. Before doing so, we note that, given a dynamic contract $\overline{\mathrm{D}}$ and a stealing strategy A, the same credit risk theory results as the ones we used in the proof of Lemma B-1.(I) imply that the manager's total expected discounted payoff at the time of his hiring, which is defined by (A.9), admits the expression

$$M(\overline{\mathrm{D}}, A) = \mathbb{E}\left[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_t \left(dC_t + \lambda\, dA_t + qG_tS_t\, dt\right)\right], \tag{B.10}$$

where $(C, S, G, \tau_{\mathrm{d}}) = \big(\overline{\mathrm{D}}_C(A), \overline{\mathrm{D}}_S(A), \overline{\mathrm{D}}_G(A), \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}(A)\big)$ and $\mathfrak{d}$ is defined by (B.5).

Definition B-2. A dynamic contract $\overline{\mathrm{D}} = (\overline{\mathrm{D}}_C, \overline{\mathrm{D}}_S, \overline{\mathrm{D}}_G, \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}, \overline{\mathrm{D}}_\beta) \in \overline{\mathcal{D}}^{w_{\mathrm{init}}}$ is *generally admissible* if

$$M(\overline{\mathrm{D}}, 0) = \max_{A\in\mathcal{A}} M(\overline{\mathrm{D}}, A) \quad \text{and} \quad M(\overline{\mathrm{D}}, 0) = \bar{w}, \tag{B.11}$$

where the manager's total expected discounted payoff M is defined by (B.10).

We denote by $\overline{\mathcal{D}}_{\mathrm{ga}}^{w_{\mathrm{init}}} \subseteq \overline{\mathcal{D}}^{w_{\mathrm{init}}}$ the family of all generally admissible contracts with initial condition $W_{0-} = w_{\mathrm{init}}$. ∎

B.3 Proof of Lemma 2

The following result establishes sufficient conditions for a dynamic contract to be generally admissible.

Lemma B-2. *A dynamic contract $\overline{\mathrm{D}} \in \overline{\mathcal{D}}^{w_{\mathrm{init}}}$ in the sense of Definition B-1 is generally admissible in the sense of Definition B-2 if*

$$w_{\mathrm{init}} = \bar{w}, \quad \overline{\mathrm{D}}_{\beta}(A) \geq \lambda \quad \text{and} \quad \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}(A) = T_W^0 \quad \text{for all } A \in \mathcal{A}, \tag{B.12}$$

and the associated solution to (B.8) for $A = 0$ satisfies the transversality condition

$$\lim_{T \to \infty} e^{-rT} \mathbb{E}\left[W_T \mathbf{1}_{\{T \leq \tau_{\mathrm{d}}\}}\right] = 0, \tag{B.13}$$

where r is the firm's discount rate.

Proof. Consider any dynamic contract $\overline{\mathrm{D}} = (\overline{\mathrm{D}}_C, \overline{\mathrm{D}}_S, \overline{\mathrm{D}}_G, \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}, \overline{\mathrm{D}}_{\beta}) \in \overline{\mathcal{D}}^{w_{\mathrm{init}}}$ and let C, S, G, τ_{d}, β be the evaluations of the contract at any given $A \in \mathcal{A}$ that are as in Definition B-1. Using (B.5), (B.8), and the integration by parts formula, we calculate

$$\mathfrak{d}_{T\wedge\tau_{\mathrm{d}}} W_{T\wedge\tau_{\mathrm{d}}} = w_{\mathrm{init}} - \int_{[0, T\wedge\tau_{\mathrm{d}}]} \mathfrak{d}_t \left(dC_t + \beta_t \, dA_t + qG_tS_t \, dt\right) + \sigma N_{T\wedge\tau_{\mathrm{d}}}, \tag{B.14}$$

where N is the stochastic integral defined by

$$N_T = \int_0^T \mathfrak{d}_t \beta_t \, dZ_t.$$

In view of (B.14) and the positivity of the stopped process $W^{\tau_{\mathrm{d}}}$, which follows from the properties of a dynamic contract, we can see that

$$0 \leq \mathfrak{d}_{T\wedge\tau_{\mathrm{d}}} W_{T\wedge\tau_{\mathrm{d}}} \leq w_{\mathrm{init}} + \sigma N_{T\wedge\tau_{\mathrm{d}}}. \tag{B.15}$$

On the other hand, Doob's L^2-inequality, Itô's isometry, (B.5) and (B.9) imply that

$$\begin{aligned} \mathbb{E}\left[\left(\sup_{T\geq 0} |N_T|\right)^2\right] &\leq 4 \sup_{T\geq 0} \mathbb{E}\left[N_T^2\right] \\ &= 4 \sup_{T\geq 0} \mathbb{E}\left[\int_0^T \mathfrak{d}_t^2 \beta_t^2 \, dt\right] \leq 4\, \mathbb{E}\left[\int_0^\infty e^{-2rt} \beta_t^2 \, dt\right] < \infty, \end{aligned} \tag{B.16}$$

therefore, N is a martingale in H^2.

Taking expectations in (B.14) and using the monotone convergence theorem, we derive the expression

$$\begin{aligned} w_{\mathrm{init}} &= \mathbb{E}\left[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_t \left(dC_t + \beta_t \, dA_t + qG_tS_t \, dt\right)\right] + \lim_{T\to\infty} \mathbb{E}\left[\mathfrak{d}_{T\wedge\tau_{\mathrm{d}}} W_{T\wedge\tau_{\mathrm{d}}}\right] \\ &= M(\overline{\mathrm{D}}, A) + \mathbb{E}\left[\int_0^{\tau_{\mathrm{d}}} \mathfrak{d}_t \left(\beta_t - \lambda\right) dA_t\right] + \lim_{T\to\infty} \mathbb{E}\left[\mathfrak{d}_{T\wedge\tau_{\mathrm{d}}} W_{T\wedge\tau_{\mathrm{d}}}\right], \end{aligned}$$

where M is the manager's total expected discounted payoff, which is defined by (B.10). In light of this calculation and the positivity of the stopped process $W^{\tau_{\rm d}}$, we can see that, if $w_{\rm init} = \bar{w}$, $\beta_t \geq \lambda$ for all $t \leq \tau_{\rm d}$, which can be true only if $\tau_{\rm d} = T^0_W$ (see requirement (b) in Definition B-1.(III)), and the transversality condition (B.13) holds true, then

$$M(\overline{\rm D}, A) \leq \bar{w} \quad \text{for all } A \in \mathcal{A}$$

and

$$M\big(\overline{\rm D}, 0\big) = \bar{w} - \lim_{T\to\infty} \mathbb{E}\left[\mathfrak{d}_{T\wedge\tau_{\rm d}} W_{T\wedge\tau_{\rm d}}\right] = \bar{w},$$

the second equality following from (B.13) and the fact that $\mathfrak{d}_T < e^{-rT}$ for all $T > 0$. We conclude that (B.12) and (B.13) are sufficient conditions for a dynamic contract to be generally admissible. ∎

In light of Lemma B-2, we henceforth focus on dynamic contracts that satisfy the requirements in (B.12) and (B.13), and refer to those as admissible dynamic contracts.

Definition B-3. An *admissible dynamic contract* is a function

$$\mathrm{D} = (\mathrm{D}_C, \mathrm{D}_S, \mathrm{D}_G, \mathrm{D}_\beta) : \mathcal{A} \to \mathcal{P}_C(0) \times \mathcal{P}_S(0) \times \mathcal{P}_G(0) \times \mathcal{P}_\beta(0)$$

such that

(I) $(\mathrm{D}_C, \mathrm{D}_S, \mathrm{D}_G, \mathrm{D}_{\tau_{\rm d}}, \mathrm{D}_\beta) \in \overline{\mathcal{D}}^{\bar{w}}$ (see Definition B-1) where

$$\mathrm{D}_{\tau_{\rm d}}(A) = \inf\{t \geq 0 \mid W_t = 0\} \in \mathcal{P}_{\tau_{\rm d}}(A) \subseteq \mathcal{P}_{\tau_{\rm d}}(0), \quad \text{for } A \in \mathcal{A},$$

in which expression W is the solution to the SDE (B.8) for

$$C = \mathrm{D}_C(A), \quad S = \mathrm{D}_S(A), \quad G = \mathrm{D}_G(A), \quad \beta = \mathrm{D}_\beta(A) \quad \text{and} \quad W_{0-} = \bar{w};$$

(II) $\mathrm{D}_\beta(A) \geq \lambda$ for all $A \in \mathcal{A}$;

(III) the solution to the SDE (B.8) for $(C, S, G, \beta) = \big(\mathrm{D}_C(0), \mathrm{D}_S(0), \mathrm{D}_G(0), \mathrm{D}_\beta(0)\big)$ and $A = 0$ satisfies the transversality condition (B.13) for $\tau_{\rm d} = \mathrm{D}_{\tau_{\rm d}}(0)$.

We denote by $\mathcal{D}$ the family of all admissible dynamic contracts. ∎

B.4 The Firm's Problem: Take Two

We now turn our attention to the firm's optimisation problem, which amounts to finding a contract $\mathrm{D}^\star \in \mathcal{D}$ that maximises the firm's expected discounted profit. In view of (A.13) and the same results from credit risk theory that we used to establish Lemma B-1.(I), we can see that, given any contract $\mathrm{D} \in \mathcal{D}$, the firm's size-adjusted expected discounted profit at the start of any manager's tenure $F(\mathrm{D})$ should satisfy

$$\begin{aligned} F(\mathrm{D}) = \mathbb{E}\bigg[&\int_0^{\tau_{\rm d}} \mathfrak{D}_t \Big(\mu + qG_t\big[(1+\gamma)F(\mathrm{D}) - \kappa - S_t\big]\Big)\, dt \\ &- \int_{[0,\tau_{\rm d}]\cap\mathbb{R}_+} \mathfrak{D}_t\, dC_t + \mathfrak{D}_{\tau_{\rm d}}\big[F(\mathrm{D}) - \kappa\big]\bigg], \end{aligned} \tag{B.17}$$

where $C = \mathrm{D}_C(0)$, $S = \mathrm{D}_S(0)$, $G = \mathrm{D}_G(0)$, $\tau_\mathrm{d} = D_{\tau_\mathrm{d}}(0)$, for D_{τ_d} as in Definition B-3.(I), and

$$\mathfrak{D}_t = \exp\left(-rt - \int_0^t qG_s\,ds\right). \tag{B.18}$$

To identify the optimal contract $\mathrm{D}^\star \in \mathcal{D}$, we first consider the following stochastic control problem.

Problem B-1. Solve the singular stochastic control problem whose value function v is defined by

$$\begin{aligned} v(w) = \sup_{(C,S,G,\beta)\in\mathcal{S}} \mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)v(\bar{w}) - \kappa - S_t\big]\Big)\,dt \\ - \int_{[0,\tau_\mathrm{d}]\cap\mathbb{R}_+} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[v(\bar{w}) - \kappa\big]\bigg], \quad \text{for } w \geq 0, \end{aligned} \tag{B.19}$$

where $\mathfrak{D}$ is defined by (B.18), $\mathcal{S}$ is the family of all control strategies (C, S, G, β) such that

$$C \in \mathcal{P}_C(0), \quad G \in \mathcal{P}_G(0), \quad S \in \mathcal{P}_S(0), \quad \beta \in \mathcal{P}_\beta(0) \text{ with } \beta \geq \lambda,$$

the associated solution to the SDE

$$dW_t = \big[\varrho W_t + qG_t(W_t - S_t)\big]\,dt - dC_t + \sigma\beta_t\,dZ_t, \quad W_{0-} = w \geq 0, \tag{B.20}$$

satisfies the transversality condition (B.13), and τ_d is the first hitting time of zero by W. ∎

Given a solution to this problem, the firm's optimisation problem reduces to solving the following one.

Problem B-2. Given an optimal control strategy $(C^\star, S^\star, G^\star, \beta^\star) \in \mathcal{S}$ for Problem B-1, determine an admissible dynamic contract $\mathrm{D}^\star \in \mathcal{D}$ such that

$$C^\star = \mathrm{D}^\star_C(0), \quad S^\star = \mathrm{D}^\star_S(0), \quad G^\star = \mathrm{D}^\star_G(0) \quad \text{and} \quad \beta^\star = \mathrm{D}^\star_\beta(0).$$

We call such an admissible dynamic contract *optimal*. ∎

The firm's expected discounted payoff at time 0 under an optimal contract $\mathrm{D}^\star$ is

$$F(\mathrm{D}^\star) = \sup_{\mathrm{D}\in\mathcal{D}} F(\mathrm{D}) = v(\bar{w}).$$

C Verification Theorem and Optimal Contract

In Appendix B, the firm's contracting problem was ultimately connected to a singular stochastic control problem (Problem B-1). The next theorem expresses the solution to this problem in terms of the solution to an appropriate HJB equation. Using this result, we characterise the solution to Problem B-2, namely, we derive the optimal admissible dynamic contract (see Theorem C-2 below). The optimality properties 1–5 stated in Section 2.1 follow immediately from these results.

Theorem C-1. *Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a concave C^2 function that satisfies the HJB equation*

$$\max\Bigg\{\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) \\ + \mu + q\big[w u'(w) - u(w) + (1+\gamma)u(\bar{w}) - \kappa\big]^+,\ -u'(w) - 1\Bigg\} = 0 \tag{C.1}$$

with the Wentzel-type boundary condition

$$u(0) = u(\bar{w}) - \kappa. \tag{C.2}$$

Define

$$w_{\rm g} = \sup\big\{w \geq 0 \mid\ w u'(w) - u(w) + (1+\gamma)u(\bar{w}) - \kappa \geq 0\big\} \vee 0 \tag{C.3}$$

and

$$w_{\rm c} = \inf\big\{w \geq 0 \mid\ u'(w) = -1\big\}, \tag{C.4}$$

with the usual conventions that $\sup\emptyset = -\infty$ and $\inf\emptyset = \infty$, and assume that $w_{\rm c} < \infty$. Furthermore, suppose that there exists a constant $K > 0$ such that

$$\big|u'(w)\big| \leq K \quad \text{for all } w > 0. \tag{C.5}$$

The following statements hold true:

(I) *If $w_{\rm g} < \infty$, then $w_{\rm g} < w_{\rm c}$.*

(II) *The function u identifies with the value function v defined by (B.19), namely,*

$$u(w) = v(w) \quad \text{for all } w \geq 0. \tag{C.6}$$

(III) *The solution $(C^\star, S^\star, G^\star, \beta^\star)$ to Problem B-1 is such that the identities*

$$S^\star_t = 0, \quad G^\star_t = \mathbf{1}_{[0, w_{\rm g}\wedge w_{\rm c}]}(W^\star_t), \quad \beta^\star_t = \lambda, \tag{C.7}$$

$$W^\star_t \in [0, w_{\rm c}] \quad \text{and} \quad C^\star_t = \int_{[0,t]} \mathbf{1}_{[w_{\rm c},\infty[}(W^\star_s)\, dC^\star_s \tag{C.8}$$

hold true for all $t \in [0, \tau^\star_{\rm d}]$, where $C^\star$, $W^\star$ are rigorously constructed as in the proof of Theorem C-2.

Proof. To show (I), we argue by contradiction and we assume that $w_{\mathrm{c}} \leq w_{\mathrm{g}} < \infty$. Combining the concavity of u with (C.1) and the definition (C.4) of w_{c}, we can see that $u'(w) = -1$ and $u(w) = u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}})$ for all $w \geq w_{\mathrm{c}}$. These observations imply that

$$u(w) - wu'(w) = u(w) + w = u(w_{\mathrm{c}}) + w_{\mathrm{c}} \quad \text{for all } w \geq w_{\mathrm{c}}.$$

In view of these identities, the assumption that $w_{\mathrm{g}} \geq w_{\mathrm{c}}$, and the definition (C.3) of w_{g}, we obtain

$$u(w) - wu'(w) = u(w_{\mathrm{g}}) + w_{\mathrm{g}} = (1+\gamma)u(\bar{w}) - \kappa \quad \text{for all } w \geq w_{\mathrm{g}},$$

which contradicts (C.3).

To show (II), we fix any initial condition $w > 0$ and any admissible control strategy $(C, S, G, \beta) \in \mathcal{S}$, where $\mathcal{S}$ is defined in the statement of Problem B-1. Using (B.20), the dynamics

$$d\mathfrak{D}_t = -(r + qG_t)\mathfrak{D}_t \, dt$$

(see (B.18)), Itô's formula and the integration by parts formula, we can see that, given any time $T > 0$,

$$\begin{aligned}
&\mathfrak{D}_{T\wedge\tau_{\mathrm{d}}} u(W_{T\wedge\tau_{\mathrm{d}}}) \\
&\quad = u(w) - \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{D}_t u'(W_{t-}) \, dC_t + \sum_{0\leq t\leq T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \big[u(W_t) - u(W_{t-}) - u'(W_{t-}) \, \Delta W_t\big] \\
&\qquad + \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \left(\frac{1}{2}\sigma^2\beta_t^2 u''(W_t) + \big[\varrho W_t + qG_t(W_t - S_t)\big]u'(W_t) - \big(r + qG_t\big)u(W_t) \right) dt \\
&\qquad + \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \sigma\beta_t u'(W_t) \, dZ_t.
\end{aligned}$$

In view of the fact that $\Delta W_t \equiv W_t - W_{t-} = -\Delta C_t$, we can see that

$$\begin{aligned}
&- \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{D}_t u'(W_{t-}) \, dC_t + \sum_{0\leq t\leq T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \big[u(W_t) - u(W_{t-}) - u'(W_{t-}) \, \Delta W_t\big] \\
&\qquad = - \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t u'(W_t) \, dC_t^{\mathrm{c}} + \sum_{0\leq t\leq T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \big[u(W_{t-} - \Delta C_t) - u(W_{t-})\big] \\
&\qquad = - \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t u'(W_t) \, dC_t^{\mathrm{c}} - \sum_{0\leq t\leq T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \int_0^{\Delta C_t} u'(W_{t-} - \Delta C_t + x) \, dx,
\end{aligned}$$

where C^{c} is the continuous part of the process C. Combining these identities with the observation that

$$u(W_{T\wedge\tau_{\mathrm{d}}}) = u(0)\mathbf{1}_{\{\tau_{\mathrm{d}}\leq T\}} + u(W_T)\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}} = \big[u(\bar{w}) - \kappa\big]\mathbf{1}_{\{\tau_{\mathrm{d}}\leq T\}} + u(W_T)\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}},$$

which follows from (C.2), we obtain

$$\int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(\bar{w}) - \kappa - S_t\big]\Big)\,dt - \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_{\mathrm{d}}}\big[u(\bar{w}) - \kappa\big]\mathbf{1}_{\{\tau_{\mathrm{d}}\le T\}}$$
$$\begin{aligned}
&= u(w) - \mathfrak{D}_T u(W_T)\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}} \\
&\quad - \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\big[u'(W_t)+1\big]\,dC_t^{\mathrm{c}} - \sum_{0\le t\le T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \int_0^{\Delta C_t} \big[u'(W_{t-} - \Delta C_t + x) + 1\big]\,dx \\
&\quad + \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\bigg(\frac{1}{2}\sigma^2\beta_t^2 u''(W_t) + \varrho W_t u'(W_t) - r u(W_t) + \mu \\
&\qquad\qquad + qG_t\Big[W_t u'(W_t) - u(W_t) + (1+\gamma)u(\bar{w}) - \kappa - S_t\big(u'(W_t)+1\big)\Big]\bigg)\,dt \\
&\quad + \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\sigma\beta_t u'(W_t)\,dZ_t.
\end{aligned}$$

The concavity of u and the fact that it satisfies the gradient constraint $u'+1\ge 0$ imply that

$$\sup_{\mathfrak{b}\ge\lambda}\big[\mathfrak{b}^2 u''(w)\big] = \lambda^2 u''(w) \quad \text{and} \quad \sup_{\mathfrak{s}\in[0,w]}\big[-\mathfrak{s}\big(u'(w)+1\big)\big] = 0.$$

Therefore, since u satisfies the HJB equation (C.1),

$$\int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(\bar{w}) - \kappa - S_t\big]\Big)\,dt - \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_{\mathrm{d}}}\big[u(\bar{w}) - \kappa\big]\mathbf{1}_{\{\tau_{\mathrm{d}}\le T\}}$$
$$\le u(w) - \mathfrak{D}_T u(W_T)\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}} + \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\sigma\beta_t u'(W_t)\,dZ_t. \tag{C.9}$$

In view of (C.5), we can see that $\big|u(w)\big| \le \big|u(0)\big| + Kw$ for all $w\ge 0$, which, combined with the transversality condition (B.13), implies that

$$\lim_{T\to\infty} \mathbb{E}\Big[\mathfrak{D}_T\big|u(W_T)\big|\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}}\Big] = 0.$$

On the other hand, we can use Itô's isometry, (B.9) and (C.5), to calculate

$$\begin{aligned}
\mathbb{E}\left[\bigg(\int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\sigma\beta_t u'(W_t)\,dZ_t\bigg)^2\right] &= \mathbb{E}\left[\int_0^{T\wedge\tau_{\mathrm{d}}} \big[\mathfrak{D}_t\sigma\beta_t u'(W_t)\big]^2\,dt\right] \\
&\le \sigma^2 K^2\,\mathbb{E}\left[\int_0^{T\wedge\tau_{\mathrm{d}}} e^{-2rt}\beta_t^2\,dt\right] \\
&< \infty,
\end{aligned}$$

which implies that the stochastic integral in (C.9) is a square-integrable martingale. In view of these results, we can take expectations in (C.9) and use the monotone convergence theorem to obtain

$$\mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(\bar{w}) - \kappa - S_t\big]\Big)\,dt - \int_{[0,\tau_\mathrm{d}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[u(\bar{w}) - \kappa\big]\bigg]$$
$$= \lim_{T\to\infty} \mathbb{E}\bigg[\int_0^{T\wedge\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(\bar{w}) - \kappa - S_t\big]\Big)\,dt$$
$$- \int_{[0,T\wedge\tau_\mathrm{d}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[u(\bar{w}) - \kappa\big]\mathbf{1}_{\{\tau_\mathrm{d}\le T\}}\bigg]$$
$$\le u(w). \tag{C.10}$$

Since $(C,S,G,\beta) \in \mathcal{S}$ has been chosen arbitrarily, it follows that

$$u(w) \ge \sup_{(C,S,G,\beta)\in\mathcal{S}} \mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(\bar{w}) - \kappa - S_t\big]\Big)\,dt$$
$$- \int_{[0,\tau_\mathrm{d}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[u(\bar{w}) - \kappa\big]\bigg]. \tag{C.11}$$

The concavity of u and the fact that this function satisfies the HJB equation (C.1) imply that

$$u'(w) = -1 \text{ for all } w \ge w_\mathrm{c}$$

and

$$wu'(w) - u(w) + (1+\gamma)u(\bar{w}) - \kappa \begin{cases} \ge 0, & \text{for all } w \in [0, w_\mathrm{g}] \\ < 0, & \text{for all } w \in\,]w_\mathrm{g}, \infty[\, \cap [0, w_\mathrm{c}] \end{cases}.$$

In view of these observations, we can check that, if $(C^\star, S^\star, G^\star, \beta^\star)$ is such that (C.7)–(C.8) hold true, then (C.9) holds with equality. We also note that this control strategy is such that the transversality condition (B.13) is satisfied because $W^\star$ takes values in the bounded interval $[0, w_\mathrm{c}]$. Following the same steps as above, we can see that (C.10) holds with equality for this control strategy, which combined with (C.11), implies that

$$u(w) = \mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}^\star} \mathfrak{D}_t^\star\Big(\mu + qG_t^\star\big[(1+\gamma)u(\bar{w}) - \kappa\big]\Big)\,dt - \int_{[0,\tau_\mathrm{d}^\star]} \mathfrak{D}_t^\star\,dC_t^\star + \mathfrak{D}_{\tau_\mathrm{d}^\star}^\star\big[u(\bar{w}) - \kappa\big]\bigg]$$
$$= \sup_{(C,S,G,\beta)\in\mathcal{S}} \mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(\bar{w}) - \kappa - S_t\big]\Big)\,dt$$
$$- \int_{[0,\tau_\mathrm{d}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[u(\bar{w}) - \kappa\big]\bigg].$$

It follows that (C.6) holds true and $(C^\star, S^\star, G^\star, \beta^\star) \in \mathcal{S}$ is optimal. ∎

The next result provides the solution to the dynamic contracting Problem B-2. The filtration $(\hat{\mathcal{F}}_t) = \big(\hat{\mathcal{F}}_t(A)\big)$ and the families $\mathcal{P}_C(A)$, $\mathcal{P}_S(A)$, $\mathcal{P}_G(A)$, $\mathcal{P}_\beta(A)$ involved in the statement of the theorem are as at the beginning of Appendix B (see (B.1)–(B.4) and (B.9) in particular).

Theorem C-2. *Suppose that the HJB equation (C.1)–(C.2) has a concave C^2 solution u such that the assumptions of Theorem C-1 are all satisfied. The following statements hold true:*

(I) *For all $A \in \mathcal{A}$, there exists a process $C = C(A) \in \mathcal{P}_C(A)$ such that, apart from a jump of size $\Delta C_0 = (\bar{w} - w_{\rm c})^+$ at time 0, C is continuous,*

$$W_t \in [0, w_{\rm c}] \quad \textit{and} \quad C_t = \int_{[0,t]} \mathbf{1}_{[w_{\rm c},\infty[}(W_s)\, dC_s \quad \textit{for all } t \geq 0, \tag{C.12}$$

where the $(\hat{\mathcal{F}}_t)$-adapted process W is the strong solution to the SDE

$$\begin{aligned} dW_t &= \Big[\varrho + q\mathbf{1}_{[0,w_{\rm g}\wedge w_{\rm c}]}(W_t)\Big] W_t\, dt - dC_t + \lambda\, (d\hat{Y}_t - \mu\, dt) \\ &= \Big[\varrho + q\mathbf{1}_{[0,w_{\rm g}\wedge w_{\rm c}]}(W_t)\Big] W_t\, dt - dC_t - \lambda\, dA_t + \sigma\lambda\, dZ_t, \quad W_{0-} = \bar{w}. \end{aligned} \tag{C.13}$$

(II) *The function $\mathrm{D}^\star = (\mathrm{D}^\star_C, \mathrm{D}^\star_S, \mathrm{D}^\star_G, \mathrm{D}^\star_\beta) : \mathcal{A} \to \mathcal{P}_C(0) \times \mathcal{P}_S(0) \times \mathcal{P}_G(0) \times \mathcal{P}_\beta(0)$ that is defined by*

$$\mathrm{D}^\star_S = 0, \quad \mathrm{D}^\star_G = \mathbf{1}_{[0,w_{\rm g}\wedge w_{\rm c}]}(W), \quad \mathrm{D}^\star_\beta = \lambda,$$

and by identifying $\mathrm{D}^\star_C(A)$ with $C(A)$ in (I) above, where W is given by (C.13), provides the solution to Problem B-2, namely, it is an optimal admissible dynamic contract.

Proof. In view of Theorem C-1 and the results asserted in (I), it is straightforward to verify that (II) is indeed true. To prove (I), we fix any $A \in \mathcal{A}$ and we recall that $\hat{Y}_t = \mu t - A_t + \sigma Z_t$. First, we assume that $w_{\rm g} = 0$, and we note that the construction corresponding to the case $w_{\rm g} = \infty$ (see also part (I) of the previous theorem) is identical if we replace ϱ by $\varrho + q$. In this case, we rewrite the SDE (C.13) in the form

$$e^{-\varrho t} W_t = \bar{w} - \hat{C}_t + \lambda \int_0^t e^{-\varrho s}\, (d\hat{Y}_s - \mu\, ds), \tag{C.14}$$

where

$$\hat{C}_t = \int_{[0,t]} e^{-\varrho s}\, dC_s.$$

Noting that

$$W_t \leq w_{\rm c} \quad \Leftrightarrow \quad -\bar{w} - \lambda \int_0^t e^{-\varrho s}\, (d\hat{Y}_s - \mu\, ds) + e^{-\varrho t} w_{\rm c} + \hat{C}_t \geq 0,$$

the analysis of Skorokhod's equation (see Lemma 6.14 in Karatzas and Shreve (1988), Chapter 3) implies that the process $\hat{C}$ defined by

$$\hat{C}_t = \sup_{\mathrm{s}\leq t} \left(\bar{w} + \lambda \int_0^{\mathrm{s}} e^{-\varrho s}\, (d\hat{Y}_s - \mu\, ds) - e^{-\varrho \mathrm{s}} w_{\rm c} \right)^+$$

is such that

$$W_t \in [0, w_c] \quad \text{and} \quad \hat{C}_t = \int_{[0,t]} \mathbf{1}_{[w_c,\infty[}(W_s)\, d\hat{C}_s \quad \text{for all } t \geq 0,$$

where W is the corresponding process in (C.14). If we define

$$C_t = \int_{[0,t]} e^{\varrho s}\, d\hat{C}_s,$$

then we can see that

$$C_t = \int_{[0,t]} e^{\varrho s}\, d\hat{C}_s = \int_{[0,t]} e^{\varrho s} \mathbf{1}_{[w_c,\infty[}(W_s)\, d\hat{C}_s = \int_{[0,t]} \mathbf{1}_{[w_c,\infty[}(W_s)\, dC_s.$$

By construction, C is $(\hat{\mathcal{F}}_t)$-adapted. Using Jensen's inequality, Doob's L^2-inequality and Itô's isometry, we calculate

$$\begin{aligned}\left(\mathbb{E}\left[\sup_{T\geq 0}\left|\int_0^T e^{-\varrho t}\, dZ_t\right|\right]\right)^2 &\leq \mathbb{E}\left[\left(\sup_{T\geq 0}\left|\int_0^T e^{-\varrho t}\, dZ_t\right|\right)^2\right] \\ &\leq 4\sup_{T\geq 0}\mathbb{E}\left[\left(\int_0^T e^{-\varrho t}\, dZ_t\right)^2\right] = 4\int_0^\infty e^{-2\varrho t}\, dt = \frac{2}{\varrho}. \end{aligned} \tag{C.15}$$

In view of this estimate, we can see that

$$\begin{aligned}\mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t}\, dC_t\right] &= \mathbb{E}\left[\lim_{T\to\infty} \hat{C}_T\right] \\ &\leq \mathbb{E}\left[\sup_{T\geq 0}\left(\bar{w} + \sigma\lambda\int_0^T e^{-\varrho t}\, dZ_t\right)^+\right] \\ &\leq \bar{w} + \sigma\lambda\,\mathbb{E}\left[\sup_{T\geq 0}\left|\int_0^T e^{-\varrho t}\, dZ_t\right|\right] \leq \bar{w} + \sigma\lambda\sqrt{\frac{2}{\varrho}}.\end{aligned}$$

It follows that $C \in \mathcal{P}_C(A)$, and C is the required process.

Before addressing the proof of (I) if $w_g \in \,]0, w_c[$, we first consider the existence and uniqueness of a strong solution to the SDE

$$dW_t = \left[\varrho + q\mathbf{1}_{[0,w_g]}(W_t)\right] W_t\, dt - dC_t - \lambda\, dA_t + \sigma\lambda\, dZ_t, \quad W_0 = \bar{w} \in \,]0, w_c[, \tag{C.16}$$

where $w_g \in \,]0, w_c[$, and $C \in \mathcal{P}_C(A)$ is continuous with $C_0 = 0$. To this end, we consider the strictly positive function

$$\begin{aligned} p_d(w) &= \exp\left(-\int_0^w \frac{2\left[\varrho + q\mathbf{1}_{[0,w_g]}(\ell)\right]\ell}{\sigma^2\lambda^2}\, d\ell\right) \\ &= \left\{\begin{array}{ll} \exp\left(-\frac{\varrho}{\sigma^2\lambda^2}w^2\right), & \text{if } w < 0 \\ \exp\left(-\frac{\varrho+q}{\sigma^2\lambda^2}w^2\right), & \text{if } w \in [0, w_g] \\ \exp\left(-\frac{\varrho}{\sigma^2\lambda^2}w^2 - \frac{q w_g^2}{\sigma^2\lambda^2}\right), & \text{if } w > w_g \end{array}\right\} \end{aligned}$$

and the strictly increasing function $p : \mathbb{R} \to \,]\underline{p}, \overline{p}[$ defined by

$$p(w) = \int_0^w p_{\mathrm{d}}(\ell)\, d\ell,$$

where

$$\underline{p} = \lim_{w \to -\infty} p(w) = -\frac{\sigma\lambda}{2}\sqrt{\frac{\pi}{\varrho}} \quad \text{and} \quad \overline{p} = \lim_{w \to \infty} p(w) \in \,]0, \infty[.$$

Using Itô's formula, we can see that, if we define $U = p(W)$, then

$$dU_t = p_{\mathrm{d}} \circ p^{-1}(U_t)\, d\Gamma_t, \quad U_0 = p(\bar{w}),$$

where

$$\Gamma_t = -C_t - \lambda A_t + \sigma\lambda Z_t \equiv -C_t - \lambda\mu t + \lambda\hat{Y}_t.$$

This SDE has a unique $(\hat{\mathcal{F}}_t)$-adapted strong solution up to the exit time of U from any interval $[\underline{u}, \overline{u}]$ such that $U_0 = p(\bar{w}) \in \,]\underline{u}, \overline{u}[$ and $[\underline{u}, \overline{u}] \subseteq \,]\underline{p}, \overline{p}[$ because Γ is a continuous $(\hat{\mathcal{F}}_t)$-semimartingale and $p_{\mathrm{d}} \circ p^{-1} : \,]\underline{p}, \overline{p}[\,\to\,]0, 1]$ is a locally Lipschitz function (see Theorem 6 in Protter (1992), Chapter V). It follows that the SDE (C.16) has a unique $(\hat{\mathcal{F}}_t)$-adapted strong solution up to the exit time of W from any bounded interval containing $W_0 = \bar{w}$. Furthermore, the expression

$$W_t = K_t \left(\bar{w} - \int_0^t K_s^{-1} \left(dC_s + \lambda dA_s - \sigma\lambda\, dZ_s \right) \right),$$

where

$$K_t = \exp\left(\int_0^t \left[\varrho + q\mathbf{1}_{[0, w_{\mathrm{g}}]}(W_s) \right] ds \right) \in \left[e^{\varrho t},\, e^{(\varrho + q)t} \right],$$

implies that the solution to the SDE (C.16) does not explode in finite time, namely, $\mathbb{P}\left(\sup_{t \in [0,T]} |W_t| < \infty \right) = 1$ for all $T \geq 0$.

We return to the proof of (I), now considering the case when $w_{\mathrm{g}} \in \,]0, w_{\mathrm{c}}[$. We first assume that $\bar{w} < w_{\mathrm{c}}$. To construct the required C, we determine a sequence of processes $(C^i,\ i \geq 0)$ and an increasing sequence of $(\hat{\mathcal{F}}_t)$-stopping times $(\nu^i,\ i \geq 0)$ such that, for all $i \geq 0$,

$$C^i \in \mathcal{P}_C(A) \quad \text{and} \quad C^i \text{ is continuous with } C_0^i = 0, \tag{C.17}$$

$$W_t^i \in [0, w_{\mathrm{c}}] \quad \text{and} \quad C_t^i = \int_{[0,t]} \mathbf{1}_{[w_{\mathrm{c}}, \infty[}(W_s^i)\, dC_s^i \quad \text{for all } t \in [0, \nu^i], \tag{C.18}$$

$$W_{\nu^i}^i \mathbf{1}_{\{\nu^i < \infty\}} = \begin{Bmatrix} w_{\mathrm{c}} \mathbf{1}_{\{\nu^i < \infty\}}, & \text{if } i \text{ is even} \\ w_{\mathrm{g}} \mathbf{1}_{\{\nu^i < \infty\}}, & \text{if } i \text{ is odd} \end{Bmatrix}, \tag{C.19}$$

$$\text{and} \quad C_t^i \mathbf{1}_{\{\nu^i < \infty\}} = C_{\nu^i}^i \mathbf{1}_{\{\nu^i < \infty\}} \quad \text{for all } t \geq \nu^i, \tag{C.20}$$

where W^i is the solution to the SDE

$$dW_t^i = \Big[\varrho + q\mathbf{1}_{[0,w_{\rm g}]}(W_t^i)\Big] W_t^i\, dt - dC_t^i + \lambda\, (d\hat{Y}_t - \mu\, dt), \quad W_0^i = \bar{w}. \tag{C.21}$$

To start with, we define

$$C^0 = 0 \quad \text{and} \quad \nu^0 = \inf\left\{t \geq 0 \mid \ W_t^0 \geq w_{\rm c}\right\},$$

and we note that (C.17)–(C.20) hold true trivially for these choices of C^0, ν^0 and for W^0 being the solution to (C.21).

Given $i \geq 0$ even and C^i, ν^i such that (C.17)–(C.20) hold true, we consider the processes

$$\hat{C}_t^{i+1} = \mathbf{1}_{\{\nu^i \leq t\} \cap \{\nu^i < \infty\}} \sup_{\mathrm{s} \in [\nu^i, t]} \left(w_{\rm c} + \lambda \int_{\nu^i}^{\mathrm{s}} e^{-\varrho(s-\nu^i)}\, (d\hat{Y}_s - \mu\, ds) - e^{-\varrho(\mathrm{s}-\nu^i)} w_{\rm c}\right)^+,$$

$$\tilde{W}_t^{i+1} = e^{\varrho(t-\nu^i)} \left(w_{\rm c} - \hat{C}_t^{i+1} + \lambda \int_{\nu^i}^{t} e^{-\varrho(s-\nu^i)}\, (d\hat{Y}_s - \mu\, ds)\right) \mathbf{1}_{\{\nu^i \leq t\} \cap \{\nu^i < \infty\}},$$

and we note that $\hat{C}^{i+1}$ is a continuous increasing process such that $\hat{C}_t^{i+1}\mathbf{1}_{\{t \leq \nu^i\}} = 0$. We then define

$$\nu^{i+1} = \inf\left\{t \geq \nu^i \mid \ \tilde{W}_t^{i+1} \leq w_{\rm g}\right\} \quad \text{and} \quad C_t^{i+1} = C_t^i + \int_0^{t \wedge \nu^{i+1}} e^{\varrho(s-\nu^i)}\, d\hat{C}_s^{i+1}.$$

In view of the analysis in the first paragraph of this proof, we can see that (C.18)–(C.20) hold true for $i+1$ in place of i. In particular, the corresponding solution W^{i+1} to (C.21) is such that $W_t^{i+1} = \tilde{W}_t^{i+1} \in \,]w_{\rm g}, w_{\rm c}]$ for all $t \in [\nu^i, \nu^{i+1}[$. We shall verify that (C.17) also holds true in the penultimate paragraph of the proof.

Given $i \geq 1$ odd and C^i, ν^i such that (C.17)–(C.20) hold true, we define $C^{i+1} = C^i$ and

$$\nu^{i+1} = \inf\{t \geq \nu^i \mid \ W_t^i \geq w_{\rm c}\}.$$

It is immediate to verify that (C.17)–(C.20) hold true for $i+1$ in place of i.

In view of the observations that

$$C_t^{i+1}\mathbf{1}_{\{t \leq \nu^i\}} = C_t^i \mathbf{1}_{\{t \leq \nu^i\}} \quad \text{for all } t \geq 0 \text{ and } i \geq 0,$$

and $\lim_{i \to \infty} \nu^i = \infty$, we can see that the required process C is given by

$$C_t = \sum_{i=1}^{\infty} C_t^i \mathbf{1}_{\{\nu^{i-1} \leq t < \nu^i\}}.$$

To see the limit invoked here, we consider the SDE

$$d\overline{W}_t^{2i} = \Big[\varrho + q\mathbf{1}_{[0,w_{\rm g}]}(\overline{W}_t^{2i})\Big] \overline{W}_t^{2i}\, dt - dC_t^{2i} - \lambda\, dA_t^{\nu^{2i-1}} + \sigma\lambda\, dZ_t, \ \overline{W}_0^{2i} = \bar{w},$$

for any $i \geq 1$, where $A_t^{\nu^{2i-1}} = A_{t \wedge \nu^{2i-1}}$, and we define

$$\overline{\nu}^{2i} = \inf\left\{t \geq \nu^{2i-1} \mid \ \overline{W}_t^{2i} \geq w_{\rm c}\right\}.$$

In view of the observations that

$$\overline{W}^{2i}_{\nu^{2i-1}}\mathbf{1}_{\{\nu^{2i-1}<\infty\}} = W^{2i}_{\nu^{2i-1}}\mathbf{1}_{\{\nu^{2i-1}<\infty\}} = w_{\mathrm{g}}\mathbf{1}_{\{\nu^{2i-1}<\infty\}},$$

which follows from the fact that

$$\overline{W}^{2i}_t\mathbf{1}_{\{t\leq\nu^{2i-1}\}} = W^{2i}_t\mathbf{1}_{\{t\leq\nu^{2i-1}\}},$$

and

$$\begin{aligned}\overline{W}^{2i}_t\mathbf{1}_{\{t\geq\nu^{2i-1}\}} &= \left(w_{\mathrm{g}} + \int_{\nu^{2i-1}}^t \left[\varrho + q\mathbf{1}_{[0,w_{\mathrm{g}}]}\big(\overline{W}^{2i}_s\big)\right]\overline{W}^{2i}_s\, ds + \sigma\lambda(Z_t - Z_{\nu^{2i-1}})\right)\mathbf{1}_{\{t\geq\nu^{2i-1}\}}\\ &\geq W^{2i}_t\mathbf{1}_{\{t\geq\nu^{2i-1}\}},\end{aligned}$$

we can see that $\overline{\nu}^{2i}\leq\nu^{2i}$ and that the strictly positive random variables $\overline{\nu}^{2i}-\nu^{2i-1}$, $i\geq 1$, are independent and identically distributed. Combining these facts with the law of large numbers, we obtain

$$\lim_{i\to\infty}\frac{\nu^{2i}}{i} > \lim_{i\to\infty}\frac{1}{i}\sum_{k=1}^{i}(\nu^{2k}-\nu^{2k-1}) \geq \lim_{i\to\infty}\frac{1}{i}\sum_{k=1}^{i}(\overline{\nu}^{2k}-\nu^{2k-1}) = \mathbb{E}\left[\overline{\nu}^2-\nu^1\right] > 0,$$

which implies that $\lim_{i\to\infty}\nu^i=\infty$, as claimed. Furthermore, given any $i\geq 1$,

$$\mathbb{E}\left[e^{-\varrho\nu^{2i}}\right] \leq \mathbb{E}\left[\prod_{k=1}^{i} e^{-\varrho(\nu^{2k}-\nu^{2k-1})}\right] = \left(\mathbb{E}\left[e^{-\varrho(\nu^2-\nu^1)}\right]\right)^i. \tag{C.22}$$

By construction, the processes C^i, $i\geq 0$, and C are all continuous, increasing and $(\hat{\mathcal{F}}_t)$-adapted. To show that these processes satisfy the integrability condition (A.4) and thus conclude that they belong to $\mathcal{P}_C(A)$ as well as that (C.17) holds true, we consider the probability spaces $(\Omega,\mathcal{F},(\mathcal{F}^i_t),\mathbb{Q}^i)$, where $(\mathcal{F}^i_t)$ are the filtrations defined by $\mathcal{F}^i_t=\mathcal{F}^Z_{\nu^{2i}+t}$ and $\mathbb{Q}^i$ are the conditional probability measures $\mathbb{P}(\cdot\mid\nu^{2i}<\infty)$ that have Radon-Nikodym derivatives with respect to $\mathbb{P}$ given by

$$\frac{d\mathbb{Q}^i}{d\mathbb{P}} = \frac{1}{\mathbb{P}(\nu^{2i}<\infty)}\mathbf{1}_{\{\nu^{2i}<\infty\}}.$$

In this context, the processes Z^i defined by $Z^i_t=(Z_{\nu^{2i}+t}-Z_{\nu^{2i}})\mathbf{1}_{\{\nu^{2i}<\infty\}}$ are standard $(\mathcal{F}^i_t)$-Brownian motions that are independent of $\mathcal{F}^i_0=\mathcal{F}^Z_{\nu^{2i}}$ (see Exercise 3.21 in Revuz and Yor (1999), Chapter IV). Furthermore,

$$\begin{aligned}\mathbb{E}&\left[\mathbf{1}_{\{\nu^{2i}<\infty\}}e^{-\varrho\nu^{2i}}\sup_{T\geq\nu^{2i}}\left|\int_{\nu^{2i}}^T e^{-\varrho(t-\nu^{2i})}\,dZ_t\right|\right]\\ &= \mathbb{P}(\nu^{2i}<\infty)\,\mathbb{E}^{\mathbb{Q}^i}\left[e^{-\varrho\nu^{2i}}\right]\mathbb{E}^{\mathbb{Q}^i}\left[\sup_{T\geq\nu^{2i}}\left|\int_{\nu^{2i}}^T e^{-\varrho(t-\nu^{2i})}\,dZ_t\right|\right]\\ &= \mathbb{E}\left[e^{-\varrho\nu^{2i}}\right]\mathbb{E}^{\mathbb{Q}^i}\left[\sup_{T\geq 0}\left|\int_0^T e^{-\varrho t}\,dZ^i_t\right|\right]\\ &\leq K\,\mathbb{E}\left[e^{-\varrho\nu^{2i}}\right],\end{aligned}$$

where $\mathbb{E}^{\mathbb{Q}^i}$ denotes expectation with respect to $\mathbb{Q}^i$ and the constant K can be determined as in (C.15). Combining such an estimate with (C.22), we calculate

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\varrho t}\, dC_t\right] \\
&\quad = \mathbb{E}\left[\sum_{i=0}^\infty \mathbf{1}_{\{\nu^{2i}<\infty\}} \int_{\nu^{2i}}^{\nu^{2i+1}} e^{-\varrho t}\, dC_t\right] \\
&\quad = \sum_{i=0}^\infty \mathbb{E}\left[\mathbf{1}_{\{\nu^{2i}<\infty\}} e^{-\varrho \nu^{2i}} \lim_{T\to\infty} \hat{C}^{2i+1}_{T\wedge \nu^{2i+1}}\right] \\
&\quad \leq \sum_{i=0}^\infty \left(w_{\rm c}\, \mathbb{E}\left[e^{-\varrho\nu^{2i}}\right] + \sigma\lambda\, \mathbb{E}\left[\mathbf{1}_{\{\nu^{2i}<\infty\}} e^{-\varrho\nu^{2i}} \sup_{T\geq \nu^{2i}} \left|\int_{\nu^{2i}}^T e^{-\varrho(t-\nu^{2i})}\, dZ_t\right|\right]\right) \\
&\quad \leq (w_{\rm c} + \sigma\lambda K) \sum_{i=0}^\infty \left(\mathbb{E}\left[e^{-\varrho(\nu^2-\nu^1)}\right]\right)^i \\
&\quad < \infty.
\end{aligned}
$$

We conclude that the processes C^i, $i \geq 0$, and C all satisfy the integrability condition (A.4) and they belong to $\mathcal{P}_C(A)$, in particular, C is the required process.

Finally, if $w_{\rm g} \in \,]0, w_{\rm c}[$ and $\bar{w} \geq w_{\rm c}$, then we can make the required construction by setting $\Delta C_0 = (\bar{w} - w_{\rm c})^+$ and then following exactly the same arguments as above simply swapping the order of considerations associated with even and odd indices. ∎

D The General Solution to a Homogeneous ODE

In this appendix, we review a range of results regarding the solvability of a second-order linear ODE on which part of our analysis has been based. All of the claims that we do not prove follow from standard theory of linear one-dimensional diffusions (e.g., see Borodin and Salminen (2002), Chapter II). To fix ideas, we consider the process

$$d\bar{W}_t = \zeta \bar{W}_t \, dt + \sigma\lambda \, dZ_t, \quad \bar{W}_0 > 0, \tag{D.1}$$

with absorption at 0, where $\zeta > 0$ is a given constant. Given any constant $\delta > 0$, there exists a pair of C^∞ functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(w_2) = \varphi(w_1) \, \mathbb{E}_{w_2}\left[e^{-\delta T_{w_1}}\right] \quad \text{for all } w_1 < w_2, \tag{D.2}$$

and

$$\psi(w_1) = \psi(w_2) \, \mathbb{E}_{w_1}\left[e^{-\delta T_{w_2}}\right] \equiv \psi(w_2) \, \mathbb{E}_{w_1}\left[e^{-\delta T_{w_2}} \mathbf{1}_{\{T_{w_2} < T_0\}}\right] \quad \text{for all } w_1 < w_2, \tag{D.3}$$

where $\mathbb{E}_{w_j}$ denotes expectation with respect to the probability measure $\mathbb{P}_{w_j}$ under which the solution to (D.1) is such that $\mathbb{P}_{w_j}(\bar{W}_0 = w_j) = 1$, for $j = 1, 2$, and T_w denotes the first hitting time of $\{w\}$, which is defined by

$$T_w = \inf\{t \geq 0 \mid \bar{W}_t = w\}, \quad \text{for } w \geq 0.$$

Note that, $\mathbb{P}_w(T_0 < \infty) > 0$ and $\mathbb{P}_w(T_\infty < \infty) = 0$ for all $w \in \,]0, \infty[$. These functions are unique, modulo multiplicative constants,

$$0 < \varphi(w) \quad \text{and} \quad \varphi'(w) < 0 \quad \text{for all } w > 0, \tag{D.4}$$

$$0 < \psi(w) \quad \text{and} \quad \psi'(w) > 0 \quad \text{for all } w > 0, \tag{D.5}$$

$$\varphi(0) = \lim_{w \downarrow 0} \varphi(w) < \infty, \quad \varphi'(0) = \lim_{w \downarrow 0} \varphi'(w) > -\infty, \tag{D.6}$$

$$\psi(0) = \lim_{w \downarrow 0} \psi(w) = 0, \quad \psi'(0) = \lim_{w \downarrow 0} \psi'(w) < \infty, \tag{D.7}$$

$$\lim_{w \to \infty} \varphi(w) = 0 \quad \text{and} \quad \lim_{w \to \infty} \psi(w) = \infty. \tag{D.8}$$

It is worth noting that (D.6)–(D.8) follow from the fact that 0 (resp., ∞) is an absorbing (resp., natural) boundary point. Furthermore, every solution to the second-order linear homogenous ODE

$$\frac{1}{2}\sigma^2\lambda^2 f''(w) + \zeta w f'(w) - \delta f(w) = 0 \tag{D.9}$$

in $]0, \infty[$ is given by

$$f(w) = A\varphi(w) + B\psi(w), \tag{D.10}$$

for some constants $A, B \in \mathbb{R}$. For future reference, we note that the fact that φ, ψ satisfy the ODE (D.9) implies that

$$\varphi = \varphi\left(\cdot\,; \frac{\zeta}{\sigma^2\lambda^2}, \frac{\delta}{\sigma^2\lambda^2}\right) \quad \text{and} \quad \psi = \psi\left(\cdot\,; \frac{\zeta}{\sigma^2\lambda^2}, \frac{\delta}{\sigma^2\lambda^2}\right),$$

namely, the two functions are parametrised by the values of $\frac{\zeta}{\sigma^2\lambda^2}$ and $\frac{\delta}{\sigma^2\lambda^2}$ only.

In the rest of our analysis, we assume that φ, ψ have been normalised through multiplication by appropriate constants so that

$$\varphi(0) = 1 \quad \text{and} \quad \psi'(0) = 1. \tag{D.11}$$

Accordingly, their Wronskian admits the expression

$$\varphi(w)\psi'(w) - \varphi'(w)\psi(w) = \exp\left(-\int_0^w \frac{2\zeta x}{\sigma^2\lambda^2}\,dx\right) = \exp\left(-\frac{\zeta}{\sigma^2\lambda^2}w^2\right). \tag{D.12}$$

In our analysis, we will make use of the following results.

Lemma D-1. *The following statements hold true.*
(I) *The function* φ *satisfies*

$$\varphi''(0) \equiv \lim_{w\downarrow 0}\varphi''(w) = \frac{2\delta}{\sigma^2\lambda^2}, \tag{D.13}$$

$$\lim_{w\downarrow 0}\big[\varphi(w) - w\varphi'(w)\big] = 1, \quad \lim_{w\to\infty}\big[\varphi(w) - w\varphi'(w)\big] = 0, \tag{D.14}$$

$$\text{and} \quad \frac{d}{dw}\big[\varphi(w) - w\varphi'(w)\big] \equiv -w\varphi''(w) < 0 \quad \text{for all } w > 0. \tag{D.15}$$

(II) *If* $\zeta > \delta$, *then*

$$\psi''(0) \equiv \lim_{w\downarrow 0}\psi''(w) = 0, \quad \lim_{w\downarrow 0}\big[\psi(w) - w\psi'(w)\big] = 0, \tag{D.16}$$

$$\text{and} \quad \frac{d}{dw}\big[\psi(w) - w\psi'(w)\big] \equiv -w\psi''(w) > 0 \quad \text{for all } w > 0. \tag{D.17}$$

Furthermore,

$$\lim_{w\downarrow 0}\frac{\psi''(w)}{\varphi''(w)} = 0, \quad \left(\frac{\psi''}{\varphi''}\right)'(w) < 0 \text{ for all } w > 0, \quad \text{and} \quad \lim_{w\to\infty}\frac{\psi''(w)}{\varphi''(w)} = -\infty. \tag{D.18}$$

Proof. The properties (D.4), (D.6), (D.8) and the fact that φ satisfies the ODE (D.9) imply immediately (D.13)–(D.15).

The limits in (D.16) follow immediately once we combine (D.7) with the fact that ψ satisfies the ODE (D.9). Since $\psi' > 0$ satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 f'''(w) + \zeta w f''(w) + (\zeta - \delta)f'(w) = 0 \tag{D.19}$$

in $]0,\infty[$, which follows from differentiating (D.9), we can see that $\psi'''(w) < 0$ for all $w > 0$ such that $\psi''(w) = 0$. Furthermore, (D.16) and (D.19) imply that

$$\psi'''(0) \equiv \lim_{w\downarrow 0}\psi'''(w) = -(\zeta - \delta)\psi'(0) = -(\zeta - \delta) < 0.$$

This inequality and (D.16) imply that $\psi''(w) < 0$ for all $w > 0$ sufficiently small. In view of these observations, a simple contradiction argument reveals that $\psi''(w) < 0$ for all $w > 0$, and (D.17) follows.

To establish (D.18), we note that the first limit follows immediately from (D.13) and (D.16). If we define $b(w) = \frac{\psi''(w)}{\varphi''(w)}$, then we can use the fact that φ, ψ satisfy the ODEs (D.9) and (D.19) to calculate

$$b'(w) = -\frac{4\delta(\zeta-\delta)}{\sigma^4\lambda^4}\exp\left(-\frac{\zeta}{\sigma^2\lambda^2}w^2\right)\big[\varphi''(w)\big]^{-2} < 0,$$

which establishes the inequality in (D.18). On the other hand, we can differentiate (D.12) to obtain

$$\begin{aligned}\varphi(w)b(w) - \psi(w) &= -\frac{2\zeta}{\sigma^2\lambda^2}w\exp\left(-\frac{\zeta}{\sigma^2\lambda^2}w^2\right)\big[\varphi''(w)\big]^{-1}\\ &= \frac{\zeta\sigma^2\lambda^2}{2\delta(\zeta-\delta)}w\varphi''(w)b'(w).\end{aligned} \tag{D.20}$$

Furthermore, we note that, since φ satisfies (D.19),

$$\lim_{w\to\infty} w\varphi''(w) = \lim_{w\to\infty}\left[-\frac{\sigma^2\lambda^2}{2\zeta}\varphi'''(w) - \frac{\zeta-\delta}{\zeta}\varphi'(w)\right] = 0. \tag{D.21}$$

We now argue by contradiction and we assume that $\lim_{w\to\infty} b(w) > -\infty$. Such an assumption implies that $\lim_{w\to\infty}\big[\varphi(w)b(w) - \psi(w)\big] = -\infty$ thanks to (D.8). This limit and (D.20) imply that $\lim_{w\to\infty} w\varphi''(w)b'(w) = -\infty$. Combining this result with (D.21), we can see that $\lim_{w\to\infty} b'(w) = -\infty$, which contradicts the assumption that $\lim_{w\to\infty} b(w) > -\infty$, and the second limit in (D.18) follows. ∎

E The High-Growth Case

In this appendix, we consider the case arising from the verification Theorem C-1 when $w_{\mathrm{g}} = w_{\mathrm{c}}$. In this context, we construct an appropriate solution to the HJB equation

$$\max\Bigg\{\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) \\ + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big],\ -u'(w) - 1\Bigg\} = 0 \tag{E.1}$$

that satisfies the Wentzel-type boundary condition

$$u(0) = u(\bar{w}) - \kappa. \tag{E.2}$$

To this end, we look for a concave C^2 function $u : \mathbb{R}_+ \to \mathbb{R}$ and a free-boundary point $w_{\mathrm{c}} > 0$ such that u satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] = 0 \tag{E.3}$$

in $]0, w_{\mathrm{c}}[$, and is given by $u(w) = u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}})$ for $w > w_{\mathrm{c}}$. In particular, we look for a solution to (E.1)–(E.2) of the form

$$u(w) = \begin{cases} A\varphi_1(w) + B\psi_1(w) + \frac{\mu + q[(1+\gamma)u(\bar{w}) - \kappa]}{r+q}, & \text{if } w \le w_{\mathrm{c}} \\ u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}}), & \text{if } w > w_{\mathrm{c}} \end{cases}, \tag{E.4}$$

for some constants $A, B \in \mathbb{R}$, where

$$\varphi_1 = \varphi_1\left(\cdot\,; \frac{\varrho + q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right) \quad \text{and} \quad \psi_1 = \psi_1\left(\cdot\,; \frac{\varrho + q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right),$$

identify with the functions φ and ψ in Appendix D for $\zeta = \varrho + q$ and $\delta = r + q$.

E.1 Analysis of the Free-Boundary Problem

We start with the case arising when it turns out that $w_{\mathrm{c}} \ge \bar{w}$. To determine the constants A, B and the free-boundary point w_{c}, we require that u should be C^2 at w_{c}, which yields the system of equations

$$u'(w_{\mathrm{c}}-) \equiv A\varphi_1'(w_{\mathrm{c}}) + B\psi_1'(w_{\mathrm{c}}) = -1 \equiv u'(w_{\mathrm{c}}+) \tag{E.5}$$

and

$$u''(w_{\mathrm{c}}-) \equiv A\varphi_1''(w_{\mathrm{c}}) + B\psi_1''(w_{\mathrm{c}}) = 0 \equiv u''(w_{\mathrm{c}}+). \tag{E.6}$$

For future reference, we observe that, given any point $w_{\mathrm{c}} > 0$, the solution to (E.3) satisfying these conditions is such that

$$G(w_{\mathrm{c}}) = 0, \tag{E.7}$$

where

$$G(w) = -(r+q)u(w) - (\varrho+q)w + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big]. \tag{E.8}$$

Also, we note that the boundary condition (E.2) implies that

$$\begin{aligned} u(0) &\equiv A + \frac{\mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big]}{r+q} \\ &= A\varphi_1(\bar{w}) + B\psi_1(\bar{w}) + \frac{\mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big]}{r+q} - \kappa \equiv u(\bar{w}) - \kappa \end{aligned} \tag{E.9}$$

(see also (D.7) and (D.11)).

Using the fact that φ_1, ψ_1 satisfy (D.9) and (D.12) in Appendix D for $\zeta = \varrho + q$ and $\delta = r + q$, we can see that the equations (E.5)–(E.6) are equivalent to

$$\begin{aligned} A &= \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_c^2\right)\left[\psi_1(w_c) - \frac{\varrho+q}{r+q}w_c\psi_1'(w_c)\right] \\ &= \frac{\sigma^2\lambda^2}{2(r+q)}\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_c^2\right)\psi_1''(w_c) < 0 \end{aligned} \tag{E.10}$$

and

$$\begin{aligned} B &= -\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_c^2\right)\left[\varphi_1(w_c) - \frac{\varrho+q}{r+q}w_c\varphi_1'(w_c)\right] \\ &= -\frac{\sigma^2\lambda^2}{2(r+q)}\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_c^2\right)\varphi_1''(w_c) < 0, \end{aligned} \tag{E.11}$$

the inequalities following thanks to the results in Lemma D-1. Substituting these expressions for A and B in (E.9), we obtain the equation

$$h(\bar{w}, w_c) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}, \tag{E.12}$$

where

$$\begin{aligned} h(\bar{w}, w) &= h\left(\bar{w}, w; \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right) \\ &= \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w^2\right)\Big[\psi_1(\bar{w})\varphi_1''(w) + \big(1 - \varphi_1(\bar{w})\big)\psi_1''(w)\Big]. \end{aligned}$$

In our analysis below, we will also need the function H defined by

$$\begin{aligned} H(\bar{w}) &= h(\bar{w}, \bar{w}) \\ &= \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}\bar{w}^2\right)\Big[\psi_1(\bar{w})\varphi_1''(\bar{w}) + \big(1 - \varphi_1(\bar{w})\big)\psi_1''(\bar{w})\Big] \\ &= \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}\bar{w}^2\right)\psi_1''(\bar{w}) + \frac{2(\varrho+q)}{\sigma^2\lambda^2}\bar{w}, \end{aligned} \tag{E.13}$$

where the last equality follows from a calculation that uses the fact that φ_1 and ψ_1 satisfy (D.9) and (D.12) for $\zeta = \varrho + q$ and $\delta = r + q$.

Lemma E-1. *There exist points* $\bar{w}_\ddagger = \bar{w}_\ddagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right)$ *and* $\bar{w}_\dagger = \bar{w}_\dagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$ *such that*

$$0 < \bar{w}_\ddagger < \bar{w}_\dagger, \quad H(\bar{w}) \begin{Bmatrix} > 0, & \text{if } \bar{w} \in]0, \bar{w}_\ddagger[\\ < 0, & \text{if } \bar{w} > \bar{w}_\ddagger \end{Bmatrix} \quad \text{and} \quad H(\bar{w}_\dagger) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}, \tag{E.14}$$

as well as a unique point $w_c = w_c\left(\bar{w}, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right) > 0$ *such that (E.12) holds true. These points are such that*

$$\bar{w} \leq \bar{w}_\dagger \;\Leftrightarrow\; w_c \geq \bar{w} \quad \text{and} \quad \bar{w} = \bar{w}_\dagger \;\Leftrightarrow\; w_c = \bar{w}. \tag{E.15}$$

Furthermore, if the problem data is such that $\bar{w} \leq \bar{w}_\dagger$, *then the function* u *defined by (E.4) for* A, B *given by (E.10), (E.11), and* w_c *being the solution to (E.12), is a concave* C^2 *solution to the HJB equation (E.1) that satisfies the boundary condition (E.2) as well as the inequality*

$$\left|u'(w)\right| \leq K \quad \text{for all } w \geq 0, \tag{E.16}$$

for some constant $K > 0$.

Proof. Differentiating h and using the fact that the functions φ_1, ψ_1 satisfy (D.19) with $\zeta = \varrho + q$ and $\delta = r + q$, we obtain

$$\frac{\partial h(\bar{w}, w)}{\partial w} = -\frac{2(\varrho - r)}{\sigma^2\lambda^2} \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2} w^2\right) \left[\psi_1(\bar{w})\varphi_1'(w) + \big(1 - \varphi_1(\bar{w})\big)\psi_1'(w)\right]$$

and

$$\frac{\partial^2 h(\bar{w}, w)}{\partial w^2} = -\frac{4(\varrho - r)(r+q)}{\sigma^4\lambda^4} \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2} w^2\right) \left[\psi_1(\bar{w})\varphi_1(w) + \big(1 - \varphi_1(\bar{w})\big)\psi_1(w)\right].$$

Recalling the properties (D.4)–(D.5), (D.8) and (D.11) of φ_1 and ψ_1, we can see that

$$\frac{\partial^2 h(\bar{w}, w)}{\partial w^2} < 0 \text{ for all } w > 0, \quad \lim_{w\to\infty} \frac{\partial^2 h(\bar{w}, w)}{\partial w^2} = -\infty \quad \text{and} \quad \lim_{w\to\infty} h(\bar{w}, w) = -\infty.$$

Combining these observations with the fact that $h(\bar{w}, 0) = \frac{2(r+q)}{\sigma^2\lambda^2}\psi_1(\bar{w}) > 0$ (see also (D.13) and (D.16)), we can see that there exists a unique $w_c > 0$ such that (E.12) holds true. Furthermore,

$$w_c \geq \bar{w} \quad \Leftrightarrow \quad H(\bar{w}) \geq -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}. \tag{E.17}$$

To show that there exist points $0 < \bar{w}_\ddagger < \bar{w}_\dagger$ such that (E.14) and (E.15) hold true, we differentiate H to obtain

$$H'(\bar{w}) = \frac{2(\varrho+q)}{\sigma^2\lambda^2} - \frac{2(\varrho - r)}{\sigma^2\lambda^2} \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}\bar{w}^2\right) \psi_1'(\bar{w})$$

and

$$H''(\bar{w}) = -\frac{4(\varrho - r)(r+q)}{\sigma^4\lambda^4} \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2}\bar{w}^2\right) \psi_1(\bar{w}).$$

It follows that H is strictly concave and $\lim_{\bar{w}\to\infty} H''(\bar{w}) = -\infty$, which implies that $\lim_{\bar{w}\to\infty} H(\bar{w}) = -\infty$. Combining these observations with the calculations

$$H(0) = 0 \quad \text{and} \quad H'(0) = \frac{2(r+q)}{\sigma^2\lambda^2} > 0$$

(see also the normalisation of φ and ψ in (D.11)), we can see that there exist unique $0 < \bar{w}_\ddagger < \bar{w}_\dagger$ such that the inequalities in (E.14) hold true. Furthermore, (E.15) follows immediately from (E.14) and (E.17).

By construction, the function u defined by (E.4) is C^2 and satisfies the boundary condition (E.2). To complete the proof, we need to show that u is concave, and that the inequalities

$$u'(w) \geq -1 \quad \text{for all } w \in [0, w_\mathrm{c}[\tag{E.18}$$

and

$$\begin{aligned} &\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) \\ &\qquad + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] \leq 0 \quad \text{for all } w > w_\mathrm{c} \end{aligned} \tag{E.19}$$

hold true.

To show that (E.18) holds with strict inequality and that the restriction of u in $]0, w_\mathrm{c}[$ is strictly concave, we define

$$\tilde{w} = \sup\big\{w \in [0, w_\mathrm{c}[\ \mid \ u'(w) \leq -1\big\} \vee 0 \in [0, w_\mathrm{c}[, \tag{E.20}$$

with the usual convention that $\sup\emptyset = -\infty$. The inequality $\tilde{w} < w_\mathrm{c}$ stated here follows immediately once we combine the boundary conditions $u''(w_\mathrm{c}) = 0$ and $u'(w_\mathrm{c}) = -1$ with the observation that $\lim_{w\uparrow w_\mathrm{c}} u'''(w) > 0$, which is true because u' satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u'''(w) + (\varrho + q)wu''(w) + (\varrho - r)u'(w) = 0$$

in $]0, w_\mathrm{c}[$. The fact that u satisfies the ODE (E.3) in $]0, w_\mathrm{c}[$ also implies that

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)w\big[u'(w) + 1\big] + G(w) = 0, \tag{E.21}$$

where G is defined by (E.8). In view of the assumption that $\varrho > r$ (see condition (1)) and the definition of $\tilde{w}$, we can see that

$$G'(w) = -(r+q)\big[u'(w) + 1\big] - (\varrho - r) < 0 \quad \text{for all } w \in [\tilde{w}, w_\mathrm{c}],$$

which, combined with (E.7), implies that $G(w) > 0$ for all $w \in [\tilde{w}, w_\mathrm{c}[$. This observation, the definition (E.20) of $\tilde{w}$, and (E.21) imply that

$$u''(w) < 0 \quad \text{for all } w \in [\tilde{w}, w_\mathrm{c}[.$$

This result and the boundary condition $u'(w_c) = -1$ imply that $u'(w) > -1$ for all $w \in [\tilde{w}, w_c[$. Combining this inequality with the definition of $\tilde{w}$ and the continuity of u', we can see that $\tilde{w} = 0$. Furthermore, (E.18) holds true with strict inequality and the restriction of u in $]0, w_c[$ is strictly concave.

Using (E.4), we can see that (E.19) is equivalent to

$$-(\varrho + q)\big[w_c + (w - w_c)\big] - (r + q)\big[u(w_c) - (w - w_c)\big] \\ + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] \leq 0 \quad \text{for all } w > w_c.$$

In view of (E.7), we note that this inequality is equivalent to $-(\varrho - r)(w - w_c) \leq 0$, which holds true because $\varrho > r$.

Finally, we note that (D.6)–(D.7) and the fact that $A, B \in \mathbb{R}$ imply that $\lim_{w \downarrow 0}\big|u'(w)\big| = \big|A\varphi'(0) + B\psi'(0)\big| < \infty$. Combining this observation with the continuity of u' and the fact that $u'(w) = -1$ for all $w \geq w_c$, we can see that (E.16) holds true. ∎

We next consider the case arising when it turns out that $w_c < \bar{w}$. In this case, C^2-continuity of the function u defined by (E.4) at w_c implies that the parameters A and B should again be given by (E.10) and (E.11). On the other hand, (E.4) and the inequality $w_c < \bar{w}$ imply that

$$\begin{aligned} u(\bar{w}) &= u(w_c) - (\bar{w} - w_c) \\ &= A\varphi_1(w_c) + B\psi_1(w_c) + \frac{\mu + q[(1+\gamma)u(\bar{w}) - \kappa]}{r + q} - (\bar{w} - w_c). \end{aligned}$$

Using this expression and (E.10)–(E.11), we can see that the boundary condition (E.2) yields the equation

$$\hat{h}(w_c, \bar{w}) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}, \tag{E.22}$$

where

$$\hat{h}(w, \bar{w}) = H(w) + \frac{2(r+q)}{\sigma^2\lambda^2}(\bar{w} - w),$$

with H being defined by (E.13).

Lemma E-2. *There exists a unique point $w_c = w_c\big(\bar{w}, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\big) > 0$ such that (E.22) holds true. Also, $w_c < \bar{w}$ if and only if $\bar{w} > \bar{w}_\dagger$, where the point $\bar{w}_\dagger = \bar{w}_\dagger\big(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\big) > 0$ is as in Lemma E-1. Furthermore, if the problem data is such that $\bar{w} > \bar{w}_\dagger$, then the function u defined by (E.4) for A, B given by (E.10), (E.11), and w_c being the solution of (E.22), is a concave C^2 solution to the HJB equation (E.1) that satisfies the boundary condition (E.2) as well as the inequality*

$$\big|u'(w)\big| \leq K \quad \textit{for all } w \geq 0,$$

for some constant $K > 0$.

Proof. In view of the analysis of the function H in the proof of Lemma E-1, we can see that $\hat{h}(\cdot, \bar{w})$ is strictly concave, $\hat{h}(0, \bar{w}) = \frac{2(r+q)}{\sigma^2\lambda^2}\bar{w} > 0$ and $\lim_{w\to\infty} \hat{h}(w, \bar{w}) = -\infty$. It follows that there exists a unique solution $w_c > 0$ to the equation (E.22). Furthermore, this solution is strictly less that $\bar{w}$ if and only if $\hat{h}(\bar{w}, \bar{w}) = H(\bar{w}) < -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}$, which is equivalent to $\bar{w} > \bar{w}_\dagger$. The rest of the proof is exactly the same as the proof of Lemma E-1. ∎

The following result provides a necessary and sufficient condition for the function u to identify with the value function.

Lemma E-3. *The concave C^2 function u studied in Lemma E-1 or Lemma E-2, depending on whether $\bar{w} \le \bar{w}_\dagger$ or not, where $\bar{w}_\dagger > 0$ is as in Lemma E-1, identifies with the value function v if and only if the problem data is such that the inequality*

$$\gamma(\mu + r\kappa) \ge (r + \gamma\varrho)w_c - \frac{\sigma^2\lambda^2 r(1+\gamma)}{2(r+q)}H(w_c) \tag{E.23}$$

holds true, where H is defined explicitly by (E.13) and w_c solves (E.12) or (E.22), depending on the case.

Proof. By construction, the function u will satisfy the HJB equation (C.1)–(C.2) in Theorem C-1 if and only if

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - ru(w) + \mu \le 0 \quad \text{for all } w \in \left]0, w_c\right[. \tag{E.24}$$

Since u satisfies the ODE (E.3), we can see that (E.24) holds true if and only if

$$-wu'(w) + u(w) - (1+\gamma)u(\bar{w}) + \kappa \le 0 \quad \text{for all } w \in \left]0, w_c\right[.$$

Furthermore, the concavity of u and (E.5) imply that this inequality is equivalent to

$$u(w_c) \le -w_c + (1+\gamma)u(\bar{w}) - \kappa, \tag{E.25}$$

which, in view of (E.7), is equivalent to

$$w_c \ge \frac{\mu + r\kappa - r(1+\gamma)u(\bar{w})}{\varrho - r}. \tag{E.26}$$

Using the identities

$$u(\bar{w}) = u(0) + \kappa = A + \frac{\mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big]}{r+q} + \kappa$$

to derive an expression for $u(\bar{w})$, and substituting for A using (E.10) and the definition of H in (E.13), we calculate

$$\begin{aligned} u(\bar{w}) &= \frac{r+q}{r-q\gamma}A + \frac{\mu + r\kappa}{r - q\gamma} \\ &= \frac{\sigma^2\lambda^2}{2(r-q\gamma)}H(w_c) - \frac{\varrho + q}{r-q\gamma} + \frac{\mu + r\kappa}{r-q\gamma} \end{aligned}$$

(note that this result is valid in the context of either Lemma E-1 or Lemma E-2). It is then a matter of simple algebraic manipulation to derive the equivalence of (E.26) and (E.23).

In view of the results derived in Lemmas E-1 and E-2, we conclude that u satisfies all of the requirements of Theorem C-1, and therefore $u = v$, if and only if the problem's parameters are such that (E.23) holds true. ∎

E.2 Proof of Proposition 2

Apart from (i) and (iii), all claims follow immediately from Lemmas E-1 and E-2. To prove statement (i), we first recall that the set of all permissible parameter values is

$$\Big\{(r,\varrho,\mu,\sigma,q,\gamma,\lambda,\kappa,\bar{w}) \in \mathbb{R}^9 \mid \ r,\varrho,\sigma,q,\gamma,\kappa > 0,\ \lambda \in (0,1], \\ \varrho > r > q\gamma \text{ and } \frac{\gamma\mu}{r} > \kappa + (1+\gamma)\bar{w}\Big\}$$

(see conditions (1)–(3)). We next fix any values of $r,\varrho,\sigma,q,\gamma,\kappa > 0$ and $\lambda \in (0,1]$ such that $\varrho > r > q\gamma$, and we note that these determine the value of $\bar{w}_\dagger = \bar{w}_\dagger\big(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\big)$ defined in Lemma E-1 (see (E.14)). Furthermore, we consider the inequality

$$\frac{\gamma\mu}{r} > \kappa + (1+\gamma)\bar{w} \tag{E.27}$$

(see condition (3)), as well as the inequality

$$\frac{\gamma\mu}{r} \geq \ell(\bar{w}), \tag{E.28}$$

where

$$\begin{aligned} \ell(\bar{w}) &= \ell(\bar{w};\, r,\varrho,\sigma,q,\gamma,\lambda,\kappa) \\ &= \kappa + (1+\gamma)\bar{w} + \frac{(\varrho - r)\gamma}{r} w_{\mathrm{c}} \\ &\quad + (1+\gamma)(w_{\mathrm{c}} - \bar{w}) - \frac{\sigma^2\lambda^2 r(1+\gamma)}{2(r+q)}\left[H(w_{\mathrm{c}}) + \frac{2\kappa(r+q)}{\sigma^2\lambda^2}\right], \end{aligned}$$

which is equivalent to (E.23) (recall that $w_{\mathrm{c}} = w_{\mathrm{c}}\big(\bar{w}, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\big)$). In view of Lemma E-3, the result will follow if we show that the set of values of $\mu, \bar{w} > 0$ for which (E.27)–(E.28) both hold true contains an open subset of $\mathbb{R}^2$. To this end, we use (E.14) and the identity $w_{\mathrm{c}}\big(\bar{w}_\dagger, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\big) = \bar{w}_\dagger$ (see (E.15)) to calculate

$$\ell(\bar{w}_\dagger) = \kappa + (1+\gamma)\bar{w}_\dagger + \frac{(\varrho - r)\gamma}{r}\bar{w}_\dagger. \tag{E.29}$$

The continuity of the functions H and $w_{\mathrm{c}}\big(\cdot, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\big)$ implies that there exists $\varepsilon_1 \in\,]0, \bar{w}_\dagger]$ such that

$$\ell(\bar{w}) < \ell(\bar{w}_\dagger) + 1 \quad \text{for all } \bar{w} \in\,]\bar{w}_\dagger - \varepsilon_1, \bar{w}_\dagger + \varepsilon_1[. \tag{E.30}$$

If we define $\varepsilon = \varepsilon_1 \wedge (1+\gamma)^{-1}$, then

$$\begin{aligned} \kappa + (1+\gamma)\bar{w} &< \kappa + (1+\gamma)\bar{w}_\dagger + (1+\gamma)\varepsilon \\ &\leq \kappa + (1+\gamma)\bar{w}_\dagger + 1 \\ &< \ell(\bar{w}_\dagger) + 1 \quad \text{for all } \bar{w} \in\,]\bar{w}_\dagger - \varepsilon, \bar{w}_\dagger + \varepsilon[. \end{aligned}$$

It follows that, given any point $(\mu, \bar{w})$ in the open set $\left]r\bigl(\ell(\bar{w}_\dagger)+1\bigr)/\gamma, \infty\right[\times \left]\bar{w}_\dagger - \varepsilon, \bar{w}_\dagger + \varepsilon\right[$, the inequalities (E.27)–(E.28) both hold true, and the proof of statement (iii) is complete.

Finally, statement (iii) follows immediately once we combine (E.14) and (E.15) in Lemma E-1 with the claim associated with (E.23) in Lemma E-3. In particular, we note for future reference that, if $\bar{w} = \bar{w}_\dagger$, then the equivalence of (E.23) and (E.25) imply that

$$(1+\gamma)u(\bar{w}) - \kappa - w_{\mathrm{c}} - u(w_{\mathrm{c}}) \geq 0 \quad \Leftrightarrow \quad \gamma\mu \geq r\kappa + (r + \gamma\varrho)\bar{w} \tag{E.31}$$

and

$$(1+\gamma)u(\bar{w}) - \kappa - w_{\mathrm{c}} - u(w_{\mathrm{c}}) = 0 \quad \Leftrightarrow \quad \gamma\mu = r\kappa + (r + \gamma\varrho)\bar{w}. \tag{E.32}$$

E.3 Proof of Proposition 3

Using p to stand for either λ or κ or q, we define

$$\Theta(p) = (1+\gamma)u(\bar{w};p) - \kappa - w_{\mathrm{c}}(p) - u\bigl(w_{\mathrm{c}}(p);p\bigr),$$

where we write $u(\cdot;p)$ and $w_{\mathrm{c}}(p)$ to stress the dependence of u and the free-boundary point w_{c} on p. We also use the notation

$$u' = \frac{\partial u}{\partial w}, \quad u'' = \frac{\partial^2 u}{\partial w^2}, \quad u_p = \frac{\partial u}{\partial p}, \quad u_p' = \frac{\partial^2 u}{\partial p\, \partial w} \quad \text{and} \quad u_p'' = \frac{\partial^3 u}{\partial p\, \partial w^2}.$$

In view of the identity $u'\bigl(w_{\mathrm{c}}(p);p\bigr) = -1$, we can see that

$$\begin{aligned}\Theta'(p) &= (1+\gamma)u_p(\bar{w};p) - \frac{\partial \kappa}{\partial p} - w_{\mathrm{c}}'(p) - \left[w_{\mathrm{c}}'(p)u'\bigl(w_{\mathrm{c}}(p);p\bigr) + u_p\bigl(w_{\mathrm{c}}(p);p\bigr)\right] \\ &= (1+\gamma)u_p(\bar{w};p) - \frac{\partial \kappa}{\partial p} - u_p\bigl(w_{\mathrm{c}}(p);p\bigr).\end{aligned}$$

If we fix any permissible values r_0, ϱ_0, μ_0, σ_0, q_0, γ_0, λ_0, κ_0, $\bar{w}_0$ of the parameters r, ϱ, μ, σ, q, γ, λ, κ, $\bar{w}$ such that

$$\bar{w}_0 = \bar{w}_\dagger\left(\frac{\varrho_0 + q_0}{\sigma_0^2\lambda_0^2}, \frac{r_0 + q_0}{\sigma_0^2\lambda_0^2}, \kappa_0\right) \quad \text{and} \quad \gamma_0\mu_0 = r_0\kappa_0 + (r_0 + \gamma_0\varrho_0)\bar{w}_0,$$

then we can see that this calculation and the identity $w_{\mathrm{c}}(p_0) = \bar{w}_0$ (see (E.15) in Lemma E-1) imply that

$$\Theta'(p_0) = \gamma u_p(\bar{w}_0;p_0) - \frac{\partial \kappa}{\partial p}. \tag{E.33}$$

Furthermore, we can see that (E.32) implies that

$$\Theta(p_0) = 0. \tag{E.34}$$

To proceed further, we recall that the C^2 function u satisfies the ODE (E.3) in $]0, w_{\mathrm{c}}[$, as well as the boundary conditions

$$u(0) = u(\bar{w}) - \kappa \quad \text{and} \quad u'(w_{\mathrm{c}}) = -1. \tag{E.35}$$

Differentiating with respect to λ, we can see that u_λ satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_\lambda''(w) + (\varrho + q)wu_\lambda'(w) - (r+q)u_\lambda(w) + \sigma^2\lambda u''(w) + q(1+\gamma)u_\lambda(\bar{w}) = 0$$

in $]0, w_{\rm c}[$, with boundary conditions

$$u_\lambda(0) = u_\lambda(\bar{w}) \quad \text{and} \quad u_\lambda'(w_{\rm c}) = 0.$$

In view of these expressions, a Feynman-Kac type of formula implies that

$$\begin{aligned} u_\lambda(w) &= \sigma^2\lambda\,\mathbb{E}\left[\int_0^\tau e^{-(r+q)t}u''(\tilde{W}_t)\,dt\right] + q(1+\gamma)\,\mathbb{E}\left[\int_0^\tau e^{-(r+q)t}\,dt\right]u_\lambda(\bar{w}) + Ju_\lambda(\bar{w}) \\ &= \sigma^2\lambda\,\mathbb{E}\left[\int_0^\tau e^{-(r+q)t}u''(\tilde{W}_t)\,dt\right] + \left(\frac{q(1+\gamma)}{r+q}(1-J) + J\right)u_\lambda(\bar{w}), \end{aligned} \tag{E.36}$$

where $\tilde{W}$ is the solution to the SDE

$$d\tilde{W}_t = (\varrho + q)\tilde{W}_t\,dt - dC_t + \sigma\lambda\,dZ_t, \quad \tilde{W}_0 = w, \tag{E.37}$$

with C reflecting $\tilde{W}$ in $w_{\rm c}$ in the negative direction, τ is the first hitting time of zero by $\tilde{W}$, and

$$J = \mathbb{E}\left[e^{-(r+q)\tau}\right]. \tag{E.38}$$

Evaluating the left-hand side of (E.36) at $w = \bar{w}$ and rearranging terms, we obtain

$$u_\lambda(\bar{w}) = \frac{(r+q)\sigma^2\lambda^2}{(1-J)(r-q\gamma)}\,\mathbb{E}\left[\int_0^\tau e^{-(r+q)t}u''(\tilde{W}_t)\,dt\right] < 0, \tag{E.39}$$

the inequality following thanks to the strict concavity of u in $]0, w_{\rm c}[$ and the fact that $J \in]0,1[$. Combining this result with (E.33)–(E.34) for p standing for λ and the equivalence stated in (E.31), we can see that, if $\lambda_0 < 1$, then $\Theta(\lambda) < 0$ and (20) fails for all $\lambda > \lambda_0$ sufficiently close to λ_0.

Differentiating (E.3) and (E.35) with respect to κ, we can see that u_κ satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_\kappa''(w) + (\varrho + q)wu_\kappa'(w) - (r+q)u_\kappa(w) + q(1+\gamma)u_\kappa(\bar{w}) - q = 0$$

in $]0, w_{\rm c}[$, with boundary conditions

$$u_\kappa(0) = u_\kappa(\bar{w}) - 1 \quad \text{and} \quad u_\kappa'(w_{\rm c}) = 0.$$

It follows that

$$\begin{aligned} u_\kappa(w) &= q\big[(1+\gamma)u_\kappa(\bar{w}) - 1\big]\,\mathbb{E}\left[\int_0^\tau e^{-(r+q)t}\,dt\right] + J\big[u_\kappa(\bar{w}) - 1\big] \\ &= q\big[(1+\gamma)u_\kappa(\bar{w}) - 1\big]\frac{1-J}{r+q} + J\big[u_\kappa(\bar{w}) - 1\big], \end{aligned}$$

where $\tilde{W}$, τ and J are as in (E.37)–(E.38). Evaluating the left-hand side of this expression at $w = \bar{w}$ and rearranging terms, we obtain

$$(1-J)\frac{r-q\gamma}{r+q}u_\kappa(\bar{w}) = -J - \frac{1-J}{r+q} < 0,$$

which implies $u_\kappa(\bar{w}) < 0$. Combining this inequality with (E.33)–(E.34) for p standing for κ and the equivalence stated in (E.31) we can see that $\Theta(\kappa) < 0$ and (20) fails for all $\kappa > \kappa_0$ sufficiently close to κ_0.

Finally, differentiating (E.3) and (E.35) with respect to q, we can see that u_q satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_q''(w) + (\varrho+q)wu_q'(w) - (r+q)u_q(w)$$
$$+ q(1+\gamma)u_q(\bar{w}) + (1+\gamma)u(\bar{w}) - \kappa + wu'(w) - u(w) = 0$$

in $]0, w_c[$, with boundary conditions

$$u_q(0) = u_q(\bar{w}) \quad \text{and} \quad u_q'(w_c) = 0.$$

Using the same arguments as above, we obtain

$$(1-J)\frac{r-q\gamma}{r+q}u_q(\bar{w}) = \mathbb{E}\left[\int_0^\tau e^{-(r+q)t} g(\tilde{W}_t)\, dt\right], \tag{E.40}$$

where $\tilde{W}$, τ and J are as in (E.37)–(E.38), and

$$g(w) = (1+\gamma)u(\bar{w}) - \kappa + wu'(w) - u(w). \tag{E.41}$$

The concavity of u implies that the function g is decreasing in w. On the other hand, the identity $u'(w_c) = -1$ and (E.34) imply that $g(w_c) = 0$. Therefore, the right-hand side of (E.40) is positive, and $u_q(\bar{w}) > 0$. This inequality and (E.33)–(E.34) for p standing for q, together with the equivalence stated in (E.31) imply that $\Theta(q) < 0$ and (20) fails for all $q < q_0$ sufficiently close to q_0.

E.4 Proof of Proposition 4

Fix any $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ in the interior of the set of permissible parameter values for which the firm is of a high-growth type. Statement (i) follows immediately from (E.12), (E.22) and the properties of the functions $h(\bar{w}, \cdot)$, $\hat{h}(\cdot, \bar{w})$ derived in the proofs of Lemmas E-1 and E-2.

To establish statement (ii), we further assume that the parameters are initially such that $\bar{w} = \bar{w}_\dagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$, so that $w_c = \bar{w}$ (see also Proposition 2). To establish the sensitivity of w_c with respect to λ, we consider equation (E.7) that w_c satisfies, namely,

$$-(r+q)u(w_c) - (\varrho+q)w_c + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] = 0. \tag{E.42}$$

Differentiating with respect to λ and using the same notation as the one introduced at the beginning of Section E.3, we obtain

$$w_c'(\lambda) = -\frac{(r+q)u_\lambda(w_c) - q(1+\gamma)u_\lambda(\bar{w})}{\varrho - r} = -\frac{r-q\gamma}{\varrho-r}u_\lambda(\bar{w}) > 0,$$

the second identity being true because $w_c = \bar{w} = \bar{w}_\dagger$ (see also (E.14)), and the strict inequality following from (E.39) and the permissibility conditions $\varrho > r > q\gamma$.

The sensitivity of w_c with respect to σ is the same as the one with respect to λ because w_c depends on either of these two parameters via the product $\sigma\lambda$.

To establish the sensitivity of w_c with respect to q and complete the proof, we differentiate (E.42) with respect to q to obtain

$$
\begin{aligned}
(\varrho - r)w_c'(q) &= -\Big((r+q)u_q(w_c) - q(1+\gamma)u_q(\bar{w}) - \big[(1+\gamma)u(\bar{w}) - \kappa - w_c - u(w_c)\big]\Big) \\
&= -\Big((r - q\gamma)u_q(\bar{w}) - \big[(1+\gamma)u(\bar{w}) - \kappa - w_c - u(w_c)\big]\Big),
\end{aligned}
$$

where the second identity follows from the fact that $w_c = \bar{w} = \bar{w}_\dagger$. Therefore,

$$
w_c'(q) = -\frac{(r - q\gamma)u_q(\bar{w}) - g(w_c)}{\varrho - r},
$$

where g is defined by (E.41). Combining the fact that g is decreasing in w, which follows from the concavity of u, with (E.40) and the definition (E.38) of J, we can see that

$$
(r - q\gamma)u_q(\bar{w}) > \frac{r+q}{1-J}\,\mathbb{E}\left[\int_0^\tau e^{-(r+q)t} g(w_c)\,dt\right] = g(w_c),
$$

and the desired inequality $w_c'(q) < 0$ follows.

E.5 The "No-Growth" Case

We close this section by considering the "no-growth" configuration that arises from the verification Theorem C-1 when $w_g = 0$ because its analysis is effectively identical to the one we have developed above. In this case, the value function v should identify with a solution u to the HJB equation

$$
\max\left\{\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - ru(w) + \mu,\ -u'(w) - 1\right\} = 0 \tag{E.43}
$$

that satisfies the boundary condition

$$
u(0) = u(\bar{w}) - \kappa. \tag{E.44}
$$

The solution to this HJB equation can be constructed in the same way as the solution to the HJB equation (E.1)–(E.2) that we derived in Section E.1. This solution will satisfy the HJB equation (C.1)–(C.2) in Theorem C-1 if and only if

$$
-wu'(w) + u(w) - (1+\gamma)u(\bar{w}) + \kappa \geq 0 \quad \text{for all } w \in\,]0, w_c[. \tag{E.45}
$$

The concavity of u implies that this inequality is satisfied if and only if it is true for $w = 0$. However, the fact that $\lim_{w\downarrow 0}\big|u'(w)\big| < \infty$ and the boundary condition (E.44) imply that, if the solution to (E.43)–(E.44) is such that (E.45) holds, then $u(\bar{w}) \leq 0$.

F The Low-Growth Case

In this appendix, we consider the case arising from the verification Theorem C-1 when $0 < w_{\mathrm{g}} < w_{\mathrm{c}}$. In this context, we address the problem of constructing a solution to the HJB equation (C.1)–(C.2) such that

$$wu'(w) - u(w) + (1+\gamma)u(\bar{w}) - \kappa \begin{cases} > 0, & \text{if } w \in [0, w_{\mathrm{g}}[\\ < 0, & \text{if } w \in]w_{\mathrm{g}}, w_{\mathrm{c}}] \end{cases}. \tag{F.1}$$

To this end, we look for a concave C^2 function $u : \mathbb{R}_+ \to \mathbb{R}$ and for strictly positive free-boundary points $w_{\mathrm{g}} < w_{\mathrm{c}}$ such that u satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] = 0 \tag{F.2}$$

in $]0, w_{\mathrm{g}}[$, the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho wu'(w) - ru(w) + \mu = 0 \tag{F.3}$$

in $]w_{\mathrm{g}}, w_{\mathrm{c}}[$, is given by $u(w) = u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}})$ for $w > w_{\mathrm{c}}$, and satisfies the Wentzel-type boundary condition

$$u(0) = u(\bar{w}) - \kappa. \tag{F.4}$$

F.1 Analysis of the Free-Boundary Problem

If it exists, then the solution to the free-boundary problem (F.2)–(F.4) is of the form

$$u(w) = \begin{cases} A_1\varphi_1(w) + B_1\psi_1(w) + \frac{\mu}{r+q} + \frac{q}{r+q}V_{\mathrm{g}}, & \text{if } w \in [0, w_{\mathrm{g}}] \\ A_2\varphi_2(w) + B_2\psi_2(w) + \frac{\mu}{r}, & \text{if } w \in]w_{\mathrm{g}}, w_{\mathrm{c}}] \\ u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}}), & \text{if } w \in]w_{\mathrm{c}}, \infty[\end{cases}, \tag{F.5}$$

where

$$V_{\mathrm{g}} = (1+\gamma)u(\bar{w}) - \kappa, \tag{F.6}$$

for some constants $A_1, B_1, A_2, B_2 \in \mathbb{R}$, where φ_1 and ψ_1 identify with the functions φ and ψ in Appendix D for $\zeta = \varrho + q$ and $\delta = r + q$, while φ_2 and ψ_2 identify with the functions φ and ψ in Appendix D for $\zeta = \varrho$ and $\delta = r$.

To determine the four parameters A_1, B_1, A_2, B_2 and the two free-boundary points w_{g}, w_{c}, we note that C^1 continuity of u at w_{g} implies

$$\begin{aligned} u(w_{\mathrm{g}}-) &\equiv A_1\varphi_1(w_{\mathrm{g}}) + B_1\psi_1(w_{\mathrm{g}}) + \frac{\mu}{r+q} + \frac{q}{r+q}V_{\mathrm{g}} \\ &= A_2\varphi_2(w_{\mathrm{g}}) + B_2\psi_2(w_{\mathrm{g}}) + \frac{\mu}{r} \equiv u(w_{\mathrm{g}}+) \end{aligned} \tag{F.7}$$

and

$$u'(w_g-) \equiv A_1\varphi_1'(w_g) + B_1\psi_1'(w_g) = A_2\varphi_2'(w_g) + B_2\psi_2'(w_g) \equiv u'(w_g+). \tag{F.8}$$

Furthermore, C^2 continuity of u at w_g and the fact that u satisfies (F.2)–(F.3) imply that

$$w_g u'(w_g) - u(w_g) + V_g = 0. \tag{F.9}$$

This equation is equivalent to $u'(w_g) = \big[u(w_g) - V_g\big]/w_g$, and can therefore be viewed as a "tangency condition" at w_g. C^2 continuity of u at w_c gives rise to the equations

$$u'(w_c-) \equiv A_2\varphi_2'(w_c) + B_2\psi_2'(w_c) = -1 \equiv u'(w_c+) \tag{F.10}$$

and

$$u''(w_c-) \equiv A_2\varphi_2''(w_c) + B_2\psi_2''(w_c) = 0 \equiv u''(w_c+). \tag{F.11}$$

For future reference, we observe that, given any point $w_c > 0$, the solution to (F.3) satisfying (F.10)–(F.11) is such that

$$-\varrho w_c - ru(w_c) + \mu = 0. \tag{F.12}$$

Finally, the boundary condition (F.4) implies that

$$u(0) \equiv A_1 + \frac{\mu}{r+q} + \frac{q}{r+q}V_g = u(\bar{w}) - \kappa \tag{F.13}$$

(see also (D.7) and (D.11)).

Using the fact that φ_2, ψ_2 satisfy (D.9) and (D.12) in Appendix D for $\zeta = \varrho$ and $\delta = r$, we can see that the equations (F.10)–(F.11) are equivalent to

$$\begin{aligned} A_2 \equiv A_2(w_c) &= \exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_c^2\right)\left[\psi_2(w_c) - \frac{\varrho}{r}w_c\psi_2'(w_c)\right] \\ &= \frac{\sigma^2\lambda^2}{2r}\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_c^2\right)\psi_2''(w_c) < 0 \end{aligned} \tag{F.14}$$

and

$$\begin{aligned} B_2 \equiv B_2(w_c) &= -\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_c^2\right)\left[\varphi_2(w_c) - \frac{\varrho}{r}w_c\varphi_2'(w_c)\right] \\ &= -\frac{\sigma^2\lambda^2}{2r}\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_c^2\right)\varphi_2''(w_c) < 0, \end{aligned} \tag{F.15}$$

the inequalities following thanks to the results in Lemma D-1. Also, we can verify that (F.7)–(F.8) are equivalent to

$$A_1 = Q_1(w_g, w_c) - \frac{q}{r+q}\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_g^2\right)\psi_1'(w_g)V_g \tag{F.16}$$

and

$$B_1 = Q_2(w_{\mathrm{g}}, w_{\mathrm{c}}) + \frac{q}{r+q} \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \varphi_1'(w_{\mathrm{g}}) V_{\mathrm{g}}, \tag{F.17}$$

where

$$\begin{aligned} Q_1(w_{\mathrm{g}}, w_{\mathrm{c}}) = {} & \frac{\sigma^2\lambda^2}{2r} \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \exp\left(\frac{\varrho}{\sigma^2\lambda^2} w_{\mathrm{c}}^2\right) \\ & \times \Big(\big[\varphi_2(w_{\mathrm{g}})\psi_1'(w_{\mathrm{g}}) - \varphi_2'(w_{\mathrm{g}})\psi_1(w_{\mathrm{g}})\big] \psi_2''(w_{\mathrm{c}}) \\ & \qquad + \big[\psi_1(w_{\mathrm{g}})\psi_2'(w_{\mathrm{g}}) - \psi_1'(w_{\mathrm{g}})\psi_2(w_{\mathrm{g}})\big] \varphi_2''(w_{\mathrm{c}}) \Big) \\ & + \frac{\mu q}{r(r+q)} \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \psi_1'(w_{\mathrm{g}}) \end{aligned} \tag{F.18}$$

and

$$\begin{aligned} Q_2(w_{\mathrm{g}}, w_{\mathrm{c}}) = {} & \frac{\sigma^2\lambda^2}{2r} \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \exp\left(\frac{\varrho}{\sigma^2\lambda^2} w_{\mathrm{c}}^2\right) \\ & \times \Big(\big[\varphi_1(w_{\mathrm{g}})\varphi_2'(w_{\mathrm{g}}) - \varphi_1'(w_{\mathrm{g}})\varphi_2(w_{\mathrm{g}})\big] \psi_2''(w_{\mathrm{c}}) \\ & \qquad - \big[\varphi_1(w_{\mathrm{g}})\psi_2'(w_{\mathrm{g}}) - \varphi_1'(w_{\mathrm{g}})\psi_2(w_{\mathrm{g}})\big] \varphi_2''(w_{\mathrm{c}}) \Big) \\ & - \frac{\mu q}{r(r+q)} \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \varphi_1'(w_{\mathrm{g}}). \end{aligned} \tag{F.19}$$

The tangency condition (F.9) gives rise to the equation

$$\begin{aligned} \lim_{w \downarrow w_{\mathrm{g}}} \big[u(w) - wu'(w)\big] & \equiv A_2\big[\varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}})\big] + B_2\big[\psi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_2'(w_{\mathrm{g}})\big] \\ & = V_{\mathrm{g}} - \frac{\mu}{r}. \end{aligned}$$

Substituting for A_2 and B_2 from (F.14)–(F.15), we obtain the identity

$$\begin{aligned} \frac{\sigma^2\lambda^2}{2r} \exp\left(\frac{\varrho}{\sigma^2\lambda^2} w_{\mathrm{c}}^2\right) \Big(& \big[\varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}})\big] \psi_2''(w_{\mathrm{c}}) \\ & - \big[\psi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_2'(w_{\mathrm{g}})\big] \varphi_2''(w_{\mathrm{c}}) \Big) = V_{\mathrm{g}} - \frac{\mu}{r}. \end{aligned} \tag{F.20}$$

On the other hand, the boundary condition (F.13), combined with (F.6) and (F.16), yields the equation

$$Q_1(w_{\mathrm{g}}, w_{\mathrm{c}}) = -\frac{\mu}{r+q} - \frac{\gamma\kappa}{1+\gamma} + \frac{1}{r+q}\left[q \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \psi_1'(w_{\mathrm{g}}) + \frac{r-q\gamma}{1+\gamma}\right] V_{\mathrm{g}}. \tag{F.21}$$

Similarly to the high-growth case that we have studied in Appendix E, any of the possibilities $\bar{w} \le w_{\rm g}$, $w_{\rm g} < \bar{w} \le w_{\rm c}$ or $\bar{w} > w_{\rm c}$ may hold true. In view of (F.5), (F.6) and (F.16)–(F.17), the case $\bar{w} \le w_{\rm g}$ is associated with the expression

$$V_{\rm g} = \frac{(r+q)\left[\varphi_1(\bar{w})Q_1(w_{\rm g}, w_{\rm c}) + \psi_1(\bar{w})Q_2(w_{\rm g}, w_{\rm c}) + \frac{\mu}{r+q} - \frac{\kappa}{1+\gamma}\right]}{\frac{r-q\gamma}{1+\gamma} + q\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_{\rm g}^2\right)\left[\varphi_1(\bar{w})\psi_1'(w_{\rm g}) - \psi_1(\bar{w})\varphi_1'(w_{\rm g})\right]}. \tag{F.22}$$

In view of (F.5), (F.6) and (F.14)–(F.15), the case $w_{\rm g} < \bar{w} \le w_{\rm c}$ is associated with the expression

$$V_{\rm g} = (1+\gamma)\left[A_2(w_{\rm c})\varphi_2(\bar{w}) + B_2(w_{\rm c})\psi_2(\bar{w}) + \frac{\mu}{r}\right] - \kappa, \tag{F.23}$$

while the case $\bar{w} > w_{\rm c}$ is associated with the expression

$$V_{\rm g} = (1+\gamma)\left[A_2(w_{\rm c})\varphi_2(w_{\rm c}) + B_2(w_{\rm c})\psi_2(w_{\rm c}) + \frac{\mu}{r} - (\bar{w} - w_{\rm c})\right] - \kappa. \tag{F.24}$$

We are thus faced with the following problem.

Problem F-0. Determine necessary and sufficient conditions on the permissible values of the parameters r, ϱ, μ, σ, q, γ, λ, κ, $\bar{w}$ such that

(I) the system of equations (F.20)–(F.22) has a solution $0 < w_{\rm g} < w_{\rm c}$ that satisfies $w_{\rm g} \ge \bar{w}$;

(II) the system of equations (F.20)–(F.21), (F.23) has a solution $0 < w_{\rm g} < w_{\rm c}$ that satisfies $w_{\rm g} < \bar{w} \le w_{\rm c}$;

(III) the system of equations (F.20)–(F.21), (F.24) has a solution $0 < w_{\rm g} < w_{\rm c}$ that satisfies $w_{\rm c} < \bar{w}$. ∎

Problem F-0 is substantially more challenging than the one we solved in Appendix E.1. In any of the three cases, one can substitute for $V_{\rm g}$ into (F.20)–(F.21) using (F.22), (F.23) or (F.24), depending on the case, and end up with a highly non-linear system of two equations for the two unknown free-boundary points $w_{\rm g} < w_{\rm c}$. Deriving necessary and sufficient conditions under which each of these three systems has a solution such that $\bar{w}$ is at the appropriate location is a most challenging exercise indeed. Instead of attempting to solve this, we have opted for a less ambitious approach: we show that the low-growth configuration *can* arise, namely, there exists a subset of the permissible parameter values of full Lebesgue measure in which the HJB equation (C.1)–(C.2) has a solution satisfying (F.1). To this end, we will need the following result.

Proposition F-1. *Assume that there exists a C^2 function $u : \mathbb{R}_+ \to \mathbb{R}$ whose restriction in $[0, w_{\rm c}]$ is strictly concave that satisfies the free-boundary problem (F.2)–(F.4) for some free-boundary points $0 < w_{\rm g} < w_{\rm c}$. The following statements hold true:*

(I) *u is given by (F.5) for A_1, B_1, A_2 and B_2 being defined by (F.14)–(F.17).*

(II) *u satisfies the HJB equation (C.1)–(C.2) as well as (F.1).*

(III) *u identifies with the value function v. Furthermore, $w_{\rm c}$ and $w_{\rm g}$ identify with the corresponding thresholds in Properties 2 and 5, respectively.*

Proof. Statement (I) follows immediately from the analysis at the beginning of this section, while Statement (III) follows from (II) and Theorem C-1.

To establish statement (II), we first note that that (F.1) holds true thanks to the identity (F.9) and the strict concavity of the restriction of u in $]0, w_{\mathrm{c}}[$. We will show that u satisfies the HJB (C.1) if we prove that the inequalities

$$u'(w) \geq -1 \quad \text{for all } w \in [0, w_{\mathrm{c}}[, \tag{F.25}$$

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) + \mu \leq 0 \quad \text{for all } w \in]0, w_{\mathrm{g}}[\cup]w_{\mathrm{c}}, \infty[\tag{F.26}$$

and

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q) w u'(w) - (r + q) u(w) + \mu + q V_{\mathrm{g}} \leq 0 \quad \text{for all } w > w_{\mathrm{g}} \tag{F.27}$$

hold true. The inequality (F.25) follows immediately from the concavity of u and the fact that $u'(w_{\mathrm{c}}) = -1$. To prove (F.26), we first note that, inside the interval $]0, w_{\mathrm{g}}[$, u satisfies the ODE (F.2), which can be rewritten

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) + \mu + q\big[w u'(w) - u(w) + V_{\mathrm{g}}\big] = 0.$$

In view of this identity, (F.9) and the concavity of u we can see that (F.26) holds true inside the interval $]0, w_{\mathrm{g}}[$. Inside the interval $]w_{\mathrm{c}}, \infty[$, (F.26) is equivalent to

$$-\varrho w - r\big[u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}})\big] + \mu \leq 0 \quad \Leftrightarrow \quad -\varrho w_{\mathrm{c}} - r u(w_{\mathrm{c}}) - (\varrho - r)(w - w_{\mathrm{c}}) + \mu \leq 0$$

(see (F.5)). Using (F.12), we see that this inequality is equivalent to $-(\varrho - r)(w - w_{\mathrm{c}}) \leq 0$, which is true because $\varrho > r$.

To establish (F.27), we first note that the inequality is equivalent to

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) + \mu + q\big[w u'(w) - u(w) + V_{\mathrm{g}}\big] \leq 0.$$

Combining the fact that u satisfies the ODE (F.3) inside $]w_{\mathrm{g}}, w_{\mathrm{c}}[$ with the fact that $w u'(w) - u(w) + V_{\mathrm{g}} < 0$ for all $w \in]w_{\mathrm{g}}, w_{\mathrm{c}}[$ (see (F.1)), we see that (F.27) is true inside $]w_{\mathrm{g}}, w_{\mathrm{c}}[$. Finally, inside $]w_{\mathrm{c}}, \infty[$, (F.27) is equivalent to

$$-(\varrho + q) w - (r + q)\big[u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}})\big] + \mu + q V_{\mathrm{g}} \leq 0$$

(see (F.5)). In view of (F.12), we see that this inequality is equivalent to

$$-(\varrho - r)(w - w_{\mathrm{c}}) + q\big[-w_{\mathrm{c}} - u(w_{\mathrm{c}}) + V_{\mathrm{g}}\big] \leq 0,$$

which is true thanks to (F.1), (F.10), and the fact that $\varrho > r$. ∎

F.2 Auxiliary Problems

We now study a pair of auxiliary problems on which the analysis of our main construction in Section F.3 relies.

Problem F-1. Given permissible values for the parameters r, ϱ, μ, σ, q, λ and constants w_{g}, V_{g}, s such that

$$w_{\mathrm{g}} > 0 \quad \text{and} \quad 0 < V_{\mathrm{g}} < \frac{\mu}{r}, \tag{F.28}$$

find a function $u_1 : [0, w_{\mathrm{g}}] \to \mathbb{R}$ that satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_1''(w) + (\varrho + q)w u_1'(w) - (r+q)u_1(w) + \mu + qV_{\mathrm{g}} = 0, \tag{F.29}$$

with boundary conditions

$$u_1(w_{\mathrm{g}}) = V_{\mathrm{g}} + s w_{\mathrm{g}} \quad \text{and} \quad u_1'(w_{\mathrm{g}}) = s. \tag{F.30}$$

∎

Problem F-2. Given permissible values for the parameters r, ϱ, μ, σ, λ and strictly positive constants w_{g}, V_{g}, find a free-boundary point $w_{\mathrm{c}} > w_{\mathrm{g}}$ and a function $u_2 : [0, w_{\mathrm{c}}] \to \mathbb{R}$ such that u_2 satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_2''(w) + \varrho w u_2'(w) - r u_2(w) + \mu = 0 \tag{F.31}$$

and the conditions

$$u_2'(w_{\mathrm{c}}) = -1, \quad u_2''(w_{\mathrm{c}}) = 0 \quad \text{and} \quad u_2(w_{\mathrm{g}}) - w_{\mathrm{g}} u_2'(w_{\mathrm{g}}) = V_{\mathrm{g}}. \tag{F.32}$$

∎

The next results are concerned with properties of the solutions to these problems.

Lemma F-1. *Problem F-1 has a unique solution. Furthermore, if $s > 0$, then the function u_1 is strictly increasing and strictly concave, and $u_1(0) < V_{\mathrm{g}}$.*

Proof. If it exists, the solution of Problem F-1 is of the form

$$u_1(w) = A_1\varphi_1(w) + B_1\psi_1(w) + \frac{\mu + qV_{\mathrm{g}}}{r+q}, \quad \text{for } w \in [0, w_{\mathrm{g}}].$$

The two boundary conditions at w_{g} provide a system of two linear equations for A_1 and B_1, which has a unique solution because its determinant is non-zero (see (D.12)). It follows that Problem F-1 has a unique solution.

In the rest of the proof, we assume that $s > 0$. To show that u_1 is strictly concave, we define

$$\hat{w} = \sup\bigl\{w \in [0, w_{\mathrm{g}}[\;\big|\; u_1'(w) \le s\bigr\} \vee 0 \in [0, w_{\mathrm{g}}[, \tag{F.33}$$

with the usual convention that $\sup \emptyset = -\infty$. Here, the inequality $\hat{w} < w_{\mathrm{g}}$ follows from the boundary condition $u_1'(w_{\mathrm{g}}) = s$ and the observation that (F.28)–(F.30) imply that

$$\lim_{w \uparrow w_{\mathrm{g}}} u_1''(w_{\mathrm{g}}) = -\frac{2}{\sigma^2 \lambda^2} \big[(\varrho - r) s w_{\mathrm{g}} + \mu - r V_{\mathrm{g}}\big] < 0.$$

In view of (F.29), the strict concavity of u_1 is equivalent to the inequality

$$(\varrho + q) w \big[u_1'(w) - s\big] - (r + q) u_1(w) + (\varrho + q) s w + \mu + q V_{\mathrm{g}} > 0 \tag{F.34}$$

holding true in $]0, w_{\mathrm{g}}[$. In the next paragraph, we show that

$$-(r + q) u_1(w) + (\varrho + q) s w + \mu + q V_{\mathrm{g}} > 0 \quad \text{for all } w \in [\hat{w}, w_{\mathrm{g}}[. \tag{F.35}$$

In view of the definition (F.33) of $\hat{w}$, this inequality implies that (F.34) is true for all $w \in [\hat{w}, w_{\mathrm{g}}[$, therefore

$$u_1''(w) < 0 \quad \text{for all } w \in [\hat{w}, w_{\mathrm{g}}[.$$

This result and the fact that $u_1'(w_{\mathrm{g}}) = s$ imply that $u_1'(w) > s$ for all $w \in [\hat{w}, w_{\mathrm{g}}[$. It follows that $\hat{w} = 0$, and u_1 is strictly concave.

To show (F.35), we note that this is equivalent to

$$u_1(w) < \frac{\varrho + q}{r + q} s w + \frac{\mu + q V_{\mathrm{g}}}{r + q} \quad \text{for all } w \in [\hat{w}, w_{\mathrm{g}}[.$$

Combining the boundary condition $u_1(w_{\mathrm{g}}) = V_{\mathrm{g}} + s w_{\mathrm{g}}$ with the fact that $u_1'(w) > s$ for all $w \in [\hat{w}, w_{\mathrm{g}}[$, we can see that $u_1(w) < V_{\mathrm{g}} + s w$ for all $w \in [\hat{w}, w_{\mathrm{g}}[$. It follows that a sufficient condition for (F.35) to be true is given by

$$\mu - r V_{\mathrm{g}} + (\varrho - r) s w > 0,$$

which holds true under our assumptions.

Finally, we note that the concavity of u_1 and the boundary condition $u_1'(w_{\mathrm{g}}) = s > 0$ imply that u_1 is strictly increasing in $[0, w_{\mathrm{g}}]$. Furthermore, the concavity of u_1 and (F.30) imply that

$$u_1(0) = u_1(w_{\mathrm{g}}) - \int_0^{w_{\mathrm{g}}} u_1'(w)\, dw < V_{\mathrm{g}} + s w_{\mathrm{g}} - s \int_0^{w_{\mathrm{g}}} dw = V_{\mathrm{g}}.$$

∎

Lemma F-2. *Problem F-2 has a solution if and only if the inequality*

$$\left(\frac{\varrho}{r} - 1\right) w_{\mathrm{g}} < \frac{\mu}{r} - V_{\mathrm{g}} \tag{F.36}$$

is true, in which case the solution is unique, and the following statements hold true:

(I) *The function u_2 is strictly concave and $u_2(0) < V_{\mathrm{g}}$. Furthermore, $u_2(0) > 0$ if*

$$1 - \frac{r}{\mu} V_{\mathrm{g}} < \varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}} \varphi_2'(w_{\mathrm{g}}). \tag{F.37}$$

(II) *There exists $\delta^* = \delta^*(r, \varrho, \sigma, \lambda, w_{\mathrm{g}}) > 0$ such that*

$$u_2'(w_{\mathrm{g}}) > 0 \quad \Leftrightarrow \quad \frac{\mu}{r} - V_{\mathrm{g}} > \delta^*. \tag{F.38}$$

Proof. By inspection, if Problem F-2 has a solution, then it is of the form

$$u_2(w) = A_2\varphi_2(w) + B_2\psi_2(w) + \frac{\mu}{r}, \quad \text{for } w \in [0, w_{\mathrm{c}}].$$

The two boundary conditions at w_{c} imply that A_2 and B_2 are given by (F.14)–(F.15). Furthermore, the tangency condition $u_2(w_{\mathrm{g}}) - w_{\mathrm{g}}u_2'(w_{\mathrm{g}}) = V_{\mathrm{g}}$ implies that, if it exists, the free-boundary point w_{c} should satisfy (F.20), namely,

$$\ell(w_{\mathrm{c}}) = V_{\mathrm{g}} - \frac{\mu}{r}, \tag{F.39}$$

where

$$\ell(w) \equiv \ell(w; r, \varrho, \sigma, \lambda, w_{\mathrm{g}}) := \frac{\sigma^2\lambda^2}{2r}\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w^2\right)\Big(\big[\varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}})\big]\psi_2''(w) \\ - \big[\psi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_2'(w_{\mathrm{g}})\big]\varphi_2''(w)\Big).$$

Differentiating the function ℓ and using the ODEs satisfied by φ_2, ψ_2 and their first derivatives, we obtain

$$\ell'(w) = -\frac{\varrho - r}{r}\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w^2\right)\Big(\big(\varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}})\big)\psi_2'(w) \\ - \big(\psi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_2'(w_{\mathrm{g}})\big)\varphi_2'(w)\Big)$$

and

$$\ell''(w) = -\frac{2(\varrho - r)}{\sigma^2\lambda^2}\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w^2\right)\Big(\big(\varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}})\big)\psi_2(w) \\ - \big(\psi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_2'(w_{\mathrm{g}})\big)\varphi_2(w)\Big).$$

An inspection of these expressions reveals that

$$\ell(w) < 0 \quad \text{and} \quad \ell'(w) < 0 \quad \text{for all } w > 0, \tag{F.40}$$

thanks to the inequalities established in Lemma D-1. Furthermore, the inequalities in Lemma D-1 imply that $\lim_{w\to\infty}\ell''(w) < 0$, which, combined with (F.40), implies that $\lim_{w\to\infty}\ell(w) = -\infty$. It follows that (F.39) has a solution $w_{\mathrm{c}} > w_{\mathrm{g}}$ if and only if $\ell(w_{\mathrm{g}}) > V_{\mathrm{g}} - \frac{\mu}{r}$. This inequality is equivalent to (F.36) thanks to the calculations

$$\ell(w_{\mathrm{g}}) = -\frac{\varrho - r}{r}w_{\mathrm{g}}\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_{\mathrm{g}}^2\right)\big[\varphi_2(w_{\mathrm{g}})\psi_2'(w_{\mathrm{g}}) - \varphi_2'(w_{\mathrm{g}})\psi_2(w_{\mathrm{g}})\big] = -\frac{\varrho - r}{r}w_{\mathrm{g}},$$

where we have used the ODE satisfied by φ_2, ψ_2 for the first identity, and (D.12) for the second identity.

To proceed further, we assume that Problem F-2 has a solution, namely, (F.36) holds true. To establish the strict concavity of u_2, we define

$$\tilde{w} = \sup\big\{w \in [0, w_{\mathrm{c}}[\mid \; u_2'(w) \le -1\big\} \vee 0 \in [0, w_{\mathrm{c}}[, \tag{F.41}$$

with the usual convention that $\sup\emptyset = -\infty$. The inequality $\tilde{w} < w_c$ stated here follows immediately once we combine the boundary conditions $u_2'(w_c) = -1$ and $u_2''(w_c) = 0$ with the observation that $\lim_{w\uparrow w_c} u_2'''(w) > 0$, which is true because u_2' satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_2'''(w) + \varrho w u_2''(w) + \varrho u_2'(w) = 0$$

in $]0, w_c[$. The fact that u_2 satisfies the ODE (F.31) in $]0, w_c[$ also implies that

$$\frac{1}{2}\sigma^2\lambda^2 u_2''(w) + \varrho w\big[u_2'(w) + 1\big] + F(w) = 0 \tag{F.42}$$

in $]0, w_c[$, where

$$F(w) = -\varrho w - r u_2(w) + \mu.$$

In view of the assumption that $\varrho > r$ (see condition (1)) and the definition of $\tilde{w}$, we can see that

$$F'(w) = -r\big[u_2'(w) + 1\big] - (\varrho - r) < 0 \quad \text{for all } w \in [\tilde{w}, w_c],$$

which, combined with (F.12), implies that $F(w) > 0$ for all $w \in [\tilde{w}, w_c[$. This observation, the definition (F.41) of $\tilde{w}$, and (F.42) imply that

$$u_2''(w) < 0 \quad \text{for all } w \in [\tilde{w}, w_c[. \tag{F.43}$$

This result and the boundary condition $u_2'(w_c) = -1$ imply that $u_2'(w) > -1$ for all $w \in [\tilde{w}, w_c[$. Combining this inequality with the definition of $\tilde{w}$ and the continuity of u_2', we can see that $\tilde{w} = 0$. In view of (F.43), it follows that u_2 is strictly concave on $]0, w_c[$.

To complete the proof of statement (I), we first note that the concavity of u_2 and (F.32) imply that

$$u_2(0) = u_2(w_g) - \int_0^{w_g} u_2'(w)\,dw < V_g + w_g u_2'(w_g) - u_2'(w_g)\int_0^{w_g} dw = V_g.$$

We also note that, in view of (F.14) and the definition of the function ℓ, (F.39) is equivalent to

$$A_2\left(\varphi_2(w_g) - w_g\varphi_2'(w_g) - \big[\psi_2(w_g) - w_g\psi_2'(w_g)\big]\frac{\varphi_2''(w_c)}{\psi_2''(w_c)}\right) = V_g - \frac{\mu}{r}.$$

Therefore

$$u_2(0) = A_2 + \frac{\mu}{r} = \left(1 - \frac{1 - \frac{rV_g}{\mu}}{\varphi_2(w_g) - w_g\varphi_2'(w_g) - \big[\psi_2(w_g) - w_g\psi_2'(w_g)\big]\frac{\varphi_2''(w_c)}{\psi_2''(w_c)}}\right)\frac{\mu}{r}$$

(see (D.7) and (D.11)). In view of (D.14)–(D.18) in Lemma D-1, we can see that

$$0 < \varphi_2(w_g) - w_g\varphi_2'(w_g) < \varphi_2(w_g) - w_g\varphi_2'(w_g) - \big[\psi_2(w_g) - w_g\psi_2'(w_g)\big]\frac{\varphi_2''(w_c)}{\psi_2''(w_c)},$$

and the sufficient condition (F.37) ensures that the inequality $u_2(0) > 0$ holds true.

To complete the proof, we still need to establish (II). To this end, we note that the expressions for A_2, B_2 given by (F.14)–(F.15) and the inequalities (D.4), (D.15) and (D.17) yield

$$u_2'(w_g) \equiv A_2(w_c)\varphi_2'(w_g) + B_2(w_c)\psi_2'(w_g) > 0 \quad \Leftrightarrow \quad \psi_2''(w_c)\varphi_2'(w_g) - \varphi_2''(w_c)\psi_2'(w_g) > 0$$
$$\Leftrightarrow \quad \frac{\psi_2''(w_c)}{\varphi_2''(w_c)} < \frac{\psi_2'(w_g)}{\varphi'(w_g)}. \tag{F.44}$$

In view of (D.4)–(D.5) and (D.18), we can see that there exists a unique point $w_c^* = w_c^*(r, \varrho, \sigma, \lambda, w_g) > 0$ such that

$$\frac{\psi_2''(w_c^*)}{\varphi_2''(w_c^*)} = \frac{\psi_2'(w_g)}{\varphi_2'(w_g)}$$

and the inequality (F.44) is satisfied if and only if $w_c > w_c^*$. Furthermore, an inspection of (F.39)–(F.40) reveals that $w_c > w_c^*$ if and only if $V_g - \frac{\mu}{r} < \ell(w_c^*)$. By construction, the point $\delta^* := -\ell(w_c^*) > 0$ is determined uniquely by r, ϱ, σ, λ and w_g. In particular, it does not depend on μ or V_g. ∎

Lemma F-3. *Fix any permissible values for the parameters r, ϱ, μ, σ, q, λ, and let w_g, V_g be strictly positive constants such that (F.36) and (F.38) both hold true, namely,*

$$\max\{(\varrho - r)w_g, r\delta^*\} < \mu - rV_g,$$

where $\delta^* = \delta^*(r, \varrho, \sigma, \lambda, w_g) > 0$ is as in Lemma F-2.(II). Given the solution w_c and u_2 to Problem F-2, if u_1 is the solution to Problem F-1 for $s = u_2'(w_g) > 0$ and we define

$$u(w) = \left\{ \begin{array}{ll} u_1(w), & \text{if } w \in [0, w_g[\\ u_2(w), & \text{if } w \in [w_g, w_c] \\ u_2(w_c) - (w - w_c), & \text{if } w > w_c \end{array} \right\}, \tag{F.45}$$

then u is C^2, strictly increasing in $[0, w_g]$, strictly concave in $[0, w_c]$, and such that $u(0) < V_g$. Furthermore, if there exists γ, κ and $\bar{w}$ such that

$$u(\bar{w}) - \kappa = u(0), \tag{F.46}$$
$$(1+\gamma)u(\bar{w}) - \kappa = V_g, \tag{F.47}$$

then u is a solution to the free-boundary problem (F.2)–(F.4) for the given values of the parameters r, ϱ, μ, σ, q, γ, λ, κ, $\bar{w}$.

Proof. In view of (F.29)–(F.31), the C^2 continuity of u at w_g follows immediately from the fact that $s = u_2'(w_g)$. The results derived in Lemmas F-1 and F-2 imply the strict concavity of u in $[0, w_c]$ as well as the facts that u is strictly increasing on $0, w_g]$ and the inequality $u(0) < V_g$. Finally, it is immediate to verify that u satisfies (F.2)–(F.4) if (F.46)–(F.47) hold true. ∎

Remark F-1. Fix any permissible values for the parameters r, ϱ, μ, σ, λ, let $w_{\rm g}$, $V_{\rm g}$ be strictly positive constants such that (F.36) holds true, and let u_2 be the corresponding solution to Problem F-2. Also, let $u_1(\cdot;q)$ be the solution to Problem F-1 for the same parameter values and for $s = u_2'(w_{\rm g})$, parametrised by $q > 0$. In view of the ODEs (F.29), (F.31) that u_1, u_2 satisfy and the identities

$$u_1(w_{\rm g}) = u_2(w_{\rm g}) \quad \text{and} \quad u_1'(w_{\rm g}) = u_2'(w_{\rm g}),$$

we can see that the function $q \mapsto u_1(0;q)$ is continuous and $\lim_{q\downarrow 0} u_1(0;q) = u_2(0)$. It follows that, given any $\varepsilon > 0$, there exists $q^*(\varepsilon) = q^*(\varepsilon; r, \varrho, \mu, \sigma, \lambda, w_{\rm g}, V_{\rm g}) > 0$ such that

$$u_1(0;q) \in \left]u_2(0) - \varepsilon, u_2(0) + \varepsilon\right[\quad \text{for all } q \in \left]0, q^*(\varepsilon)\right[.$$

■

F.3 Proof of Proposition 5

In view of Proposition F-1, we need to show that the set of permissible parameter values for which the free-boundary problem (F.2)–(F.4) has a C^2 strictly concave solution has non-empty interior in $\mathbb{R}^9$. To prove that this is indeed the case, we rely on Lemma F-3. Our constructive argument proceeds in two steps.

Step 1. We fix any

$$\varrho > r > 0, \quad \sigma > 0, \quad \lambda \in \left]0,1\right] \quad \text{and} \quad w_{\rm g} > 0.$$

Also, given any

$$\xi \in \left]0,1\right[\quad \text{such that} \quad \xi < \varphi_2(w_{\rm g}) - w_{\rm g}\varphi_2'(w_{\rm g})$$

(see (D.14)–(D.15)), we fix any μ such that

$$\mu > \max\left\{\frac{(\varrho - r)w_{\rm g}}{\xi}, \frac{r\delta^*}{\xi}\right\} > 0,$$

where $\delta^* = \delta^*(r, \varrho, \sigma, \lambda, w_{\rm g}) > 0$ is as in Lemma F-2, and we define

$$V_{\rm g} = (1 - \xi)\frac{\mu}{r} > 0.$$

For such choices of parameter values, Lemma F-2 implies that the solution u_2 to Problem F-2 exists, is unique, and satisfies

$$0 < u_2(0) < V_{\rm g} \quad \text{and} \quad u_2'(w_{\rm g}) > 0.$$

Furthermore, we fix any $\varepsilon \in \left]0, u_2(0)\right[$, any q such that

$$0 < q < \min\left\{q^*(\varepsilon), r\frac{u_2(0) - \varepsilon}{V_{\rm g} - u_2(0) + \varepsilon}\right\}, \tag{F.48}$$

where $q^*(\varepsilon) > 0$ is as in Remark F-1, and we let u_1 be the unique corresponding solution to Problem F-1 for $s = u_2'(w_\mathrm{g}) > 0$. In view of Lemma F-3 and Remark F-1, we can see that, for such choices,

$$\begin{gathered}\text{the function } u \text{ defined by (F.45) is } C^2,\\ \text{strictly increasing in } [0, w_\mathrm{g}], \text{ strictly concave in } [0, w_\mathrm{c}],\\ \text{and such that } u(0) \in \,]0, V_\mathrm{g}[\text{ and } u(0) \in \,]u_2(0) - \varepsilon, u_2(0) + \varepsilon[.\end{gathered} \tag{F.49}$$

Step 2. Given parameter values as in the previous step, the function u defined by (F.45) will satisfy (F.46)–(F.47) and provide a solution to the free-boundary problem (F.2)–(F.4) if

$$\kappa = u(\bar{w}) - u(0) \quad \text{and} \quad \gamma = \frac{V_\mathrm{g} - u(0)}{u(\bar{w})}. \tag{F.50}$$

The analysis of the previous step and the continuity of all functions involved in it imply that there exists an open subset of $\mathbb{R}^8$ such that (F.49) holds true for every choice of $(r, \varrho, \mu, \sigma, q, \lambda, w_\mathrm{g}, V_\mathrm{g})$ in this set. In view of this observation, the construction will be complete if we show that there exists a continuous function $\bar{w}_*$ mapping this set into $]0, \frac{\mu}{r}[$ such that the parameters κ, γ defined by (F.50) are strictly positive and Conditions (2) and (3), which are equivalent to

$$\frac{\kappa + \bar{w}}{\frac{\mu}{r} - \bar{w}} \equiv \frac{u(\bar{w}) - u(0) + \bar{w}}{\frac{\mu}{r} - \bar{w}} < \frac{V_\mathrm{g} - u(0)}{u(\bar{w})} \equiv \gamma < \frac{r}{q} \tag{F.51}$$

in the present context, are satisfied for all $\bar{w} \in \,]0, \bar{w}_*[$.

Since u is strictly increasing in $[0, w_\mathrm{g}]$, the strict positivity of κ defined in (F.50) follows immediately as long as $\bar{w} < w_\mathrm{g}$. On the other hand, given any $\bar{w} \in \,]0, w_\mathrm{g}[$, the strict positivity of γ defined in (F.50) follows from the inequalities $u(0) < V_\mathrm{g}$ and $u(\bar{w}) > u(0) > 0$ (see (F.49)). We therefore need to determine $\bar{w}_* \in \left]0, \min\{w_\mathrm{g}, \frac{\mu}{r}\}\right[$ such that (F.51) holds true for all $\bar{w} \in \,]0, \bar{w}_*[$. To this end, we note that the second inequality in (F.51) follows from the inequalities

$$\frac{V_\mathrm{g} - u(0)}{u(\bar{w})} < \frac{V_\mathrm{g} - u(0)}{u(0)} < \frac{V_\mathrm{g} - u_2(0) + \varepsilon}{u_2(0) - \varepsilon} < \frac{r}{q},$$

which hold true for all $\bar{w} < w_\mathrm{g}$ thanks to (F.48) and (F.49). Finally, the existence of the required $\bar{w}_* \in \left]0, \min\{w_\mathrm{g}, \frac{\mu}{r}\}\right[$ follows from the fact that the first inequality in (F.51) holds true for $\bar{w} = 0$ and the continuity of u.

G Data Sources and Variable Construction

Data Sources. Our sample relies on information on CEO turnover and CEO compensation as reported in the widely used Standard and Poors ExecuComp database from January 1992 to December 2013. Accounting information comes from the Compustat Industrial Annual files, and stock price and stock return information comes from the monthly CRSP tapes. The dataset is at annual frequency (on a calendar year basis), although our measure of past performance is constructed using stock return data at monthly frequency.

CEO Episodes and Turnover. The starting point of the construction of the data set is to identify CEO episodes, which track the tenure of a given manager as CEO of a given firm. Using the information available in ExecuComp, we define the first year of a CEO episode as the first year in which the CEO is reported as being in charge of the firm. The variable *TenureYear* is set at 1 in the year of his appointment and is incremented for each calendar year he remains in office. A turnover event is recorded in the year that ExecuComp reports the CEO leaves office. In cases where ExecuComp does not report a date leaving office but a new CEO is reported for the same firm in a subsequent year, a turnover event is recorded in the last year of the old CEO's reported tenure. The variable *Turnover* is a binary variable which equals 1 for a CEO episode in the year of a turnover event and zero otherwise. The variable *TotTenure* equals the total number of years the CEO runs the firm.

Compensation. We define the variable *TotPay* as the total annual compensation as recorded in the ExecuComp variable `tdc1`. This includes salary, cash bonus, retirement benefits, stock, and stock options in the year they are awarded.

Average Q and Growth-Related Proxies. For a given CEO episode, the variable *IndQInit* is equal to the arithmetic mean of the 'average Q' of all firms in the same 4-digit SIC industry group in the year before the CEO was appointed. Average Q is defined as the ratio of the market value of assets divided by the book value of assets (`at`). The market value of assets is equal to total assets (`at`), plus the product of common stock holdings (`csho`) times the closing stock price at the end of the fiscal year (`prcc_c`), minus the book value of common equity (`ceq`). Consistent with Almeida and Campello (2007), we set as missing those values of Q above 10. In constructing the mean industry average Q, we consider the SIC industry definition as reported in Compustat. Our proxy for the arrival of a growth opportunity, *RatioQ*, is equal to the ratio of the mean industry average Q in a given year to *IndQinit*, i.e., the mean industry average Q the year before the CEO was appointed.

Cumulative Abnormal Returns. In the regressions reported in the main text, we measure past performance using 2-year annualized cumulative abnormal stock returns, which we denote by CAR. When evaluated in year t, the variable CAR corresponds to the annualized cumulative abnormal return between January of year $t-1$ and December of year t. The results presented in the paper are robust to lagging the performance measure by one year (i.e., measuring annualized cumulative abnormal return between Jan of year $t-2$ and Dec of year $t-1$) and to the use of shorter or longer window lengths for the measurement of past performance. To construct our CAR variable, we use monthly return data from CRSP to obtain abnormal returns at monthly frequency, compute compounded cumulative abnormal returns over 24 months, and annualize. To obtain monthly abnormal returns, we compute Dimson betas using rolling regressions over a 60-month time window, where the explanatory variables in the regressions include the current, lagged and forward values of the return on the market portfolio proxied by the CRSP value-weighted index.

Other Controls. The variable *lnAssets* equals the logarithm of the total assets of the firm as reported in Compustat (`at`). The variable *ROA*, or return on assets, equals the ratio of earnings (`ib`) over total assets (`at`) as reported in Compustat. All variables are winsorized at the 1% level.

Compensation Duration. The variable *PayDuration* is computed according to the duration formula (25) displayed in the main text. The sample of CEO episodes for which this variable is computed comprises: (i) all completed episodes for which we observe annual compensation from year 1 of the CEO's tenure until he leaves post; and (ii) episodes in which the CEO is still in office by the end of our sample period but has been managing the firm for at least 15 years, with no interruption in reported compensation. The results presented in Section 4.3 of the paper are qualitatively unaffected when the latter group of episodes is removed from the sample.